Enclosing Solutions of Integral Equations

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The subject of this talk is integral operators of the type

$$Ty(x) = f(x) + \int_a^b g(x,t,y(x),y(t))dt, \qquad (1)$$

where $x \in [a,b]$ which is a real interval, and f, g are given smooth functions. The set of functions considered is the set of all functions y for which the integral of (1) exists. It is denoted by $\mathcal{F}([a,b])$.

We wish to solve the integral equation

$$Ty = y, \qquad (2)$$

and provide guaranteed bounds for a solution. That is, we wish to find a narrow function interval $Y^\star$ which is guaranteed to contain a solution $y^\star$ of (1). A *function interval* Y is an interval of functions in $\mathcal{F}([a,b])$,

$$Y = [\underline{y}, \overline{y}] =$$
$$\{y \in \mathcal{F}([a,b]) | \forall t \in [a,b] : \underline{y}(t) \le y(t) \le \overline{y}(t)\}$$

where $\underline{y}, \overline{y} \in \mathcal{F}([a,b])$.

The basic method. In order to find $Y^\star$ we use iterated sets: For any function interval Y we calculate an enclosure ITY of

$$TY = \{Ty \mid y \in Y\}$$

Then the iteration

$$Y^{(k+1)} = Y^{(k)} \cap ITY^{(k)} \quad k = 0, 1, \ldots \qquad (3)$$

defines a nested sequence of function intervals with the following properties (see [2]):

$$y^\star \in Y^{(k)} \quad \Rightarrow \quad y^\star \in Y^{(k+1)} \qquad (4)$$
$$Y^{(k+1)} = \emptyset \quad \Rightarrow \quad \text{no solution in } Y^{(k)} \qquad (5)$$
$$ITY^{(k)} \subseteq Y^{(k)} \quad \Rightarrow \quad \text{solution exists in } ITY^{(k)} \qquad (6)$$

Mean value forms. We base our practical version of (3) on the mean value form of T, [3],

$$T_m Y = Ty_m + (T'Y)(Y - y_m) \qquad (7)$$

where the real function y_m is chosen as the "lower bound" of Y,

$$y_m(x) = \underline{y}(x). \qquad (8)$$

More detailed $T_m Y$ is the following, [3, 9],

$$T_m Y(x) = Ty_m(x) \qquad (9)$$
$$+ (Y(x) - y_m(x)) \int_a^b G_3'(x,t,Y(x),Y(t))dt$$
$$+ \int_a^b G_4'(x,t,Y(x),Y(t))(Y(t) - y_m(t))dt$$

where G_3' and G_4' are interval extensions of the partial derivatives of g with respect to its third and fourth variable. Interval integrals are defined in [4].

Now the integrals are enclosed by Riemann sums in order to obtain ITY. ITY is from now on denoted by IT_mY. The subdivision used is denoted

$$X_k = [x_k, x_{k+1}] \quad , k = 0, \ldots, n-1 \qquad (10)$$

where $x_0 = a$ and $x_n = b$. Then IT_mY is the following,

$$IT_m Y(x) = Ty_m(x) \qquad (11)$$
$$+ (Y(x) - y_m(x)) \sum_{j=0}^{n-1} G_3'(X_i, X_j, Y_i, Y_j)\mathrm{w}(X_j)$$
$$+ \sum_{j=0}^{n-1} G_4'(X_i, X_j, Y_i, Y_j) \int_{X_j} (Y(t) - y_m(t))dt,$$

where $x \in X_i$ and $Y_k = Y(X_k)$ is the smallest real interval containing the range of Y over X_k, $\{Y(x) \mid x \in X_k\}$. The only practical problem left here is calculation of the non-interval term $Ty_m(x)$.

It can be shown that under reasonable assumptions the distance between IT_mY and T_mY is of the order $\mathrm{O}(\mathrm{w}(Y)\max(\Delta, \mathrm{w}(Y)))$ where $\Delta = \max(\mathrm{w}(X_i))$. Furthermore it can be shown that a contraction property of T is not wasted by extending T to its mean value form and enclosing the interval integrals by Riemann sums. We can still obtain $Y^{(k)} \to [y^\star, y^\star]$ for $\Delta \to 0$. Therefore, the only source of error

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will be the non-interval term in (11): $Ty_m(x)$ (and rounding errors when it comes to real calculations).

Finally we describe how Ty_m is enclosed. We give two different enclosions, using zero and first order outer approximations, respectively.

Zero order approximation:

If $x \in X_i$ then $Ty_m \subseteq I_0T\ y_m$ where

$$I_0T\ y_m(x) = F(X_i) + \sum_{j=0}^{n-1} G(X_i, X_j, Y_{m,i}, Y_{m,j}) \mathrm{w}(X_j) \tag{12}$$

and

$$Y_{m,k} = [\inf_{t \in X_k} (y_m(t)), \sup_{t \in X_k} (y_m(t))].$$

where F and G are interval extensions of f and g.

Thus the final enclosure of TY is (11) with the enclosure (12) of Ty_m inserted. Notice that if Y is bounded by piecewise constant functions then the same holds for the enclosure.

First order approximation:

Since y_m is a real function (not a variable) when Ty_m is calculated we can consider g as a function of x and t only,

$$h(x,t) = g(x, t, y_m(x), y_m(t)). \tag{13}$$

If $x \in X_i$ then $Ty_m \subseteq I_1T\ y_m$ where

$$I_1T\ y_m(x) = f(x_i) + F'(X_i)(x - x_i) + \sum_{j=0}^{n-1} \Big\{ (h(x_i, x_j) + H_1'(X_i, X_j)(x - x_i)) \mathrm{w}(X_j) + H_2'(X_i, X_j) \cdot \frac{1}{2} \mathrm{w}(X_j)^2 \Big\} \tag{14}$$

where F', H_1' and H_2' are interval extensions of the gradients f', h_1' and h_2'.

Thus the final enclosure of TY is (11) with the enclosure (14) of Ty_m inserted. Notice that if Y is bounded by piecewise linear functions then the same holds for the enclosure.

Assume T is an interval contraction. Under reasonable assumptions we then obtain convergence to a function interval $Y^\star$ of the width $\mathrm{O}(\Delta)$ for the zero order approximation, and $\mathrm{O}(\Delta^2)$ for the first order approximation, where Δ is $\max(\mathrm{w}(X_i))$.

A numerical example. For the numerical calculations we have used the C++ interval package PROFIL [5].

Consider the integral equation [8],

$$y(x) = 1 + \frac{\lambda}{2} \int_0^1 \frac{x}{x+t} y(x) y(t) dt, \tag{15}$$

where $\lambda \in [0,1]$. The results shown in the following tables correspond to $\lambda = \frac{1}{2}$. Then T is a contraction. Notice that our method also allows an interval to be inserted for λ.

i	$n = 15$	$n = 30$	$n = 60$
0	0.41000	0.41000	0.41000
1	0.21141	0.16021	0.13589
2	0.14997	0.08580	0.05631
3	0.13061	0.06556	0.03552
4	0.12437	0.06012	0.03040
5	0.12232	0.05864	0.02912
6	0.12164	0.05824	0.02880
7	0.12141	0.05813	0.02872
8	0.12133	0.05810	0.02870
9	0.12130	0.05809	0.02869
...	...	...	...
∞	0.12129 (34)	0.05809 (30)	0.02869 (29)

Table 1: Zero order approximation: The width of $Y^{(i)}$ as a function of the iteration number i using $n = 15, 30, 60$ discretization intervals. The number in parenthesis is the total number of iterations before the algorithm stops (when $Y^{(i)}$ becomes invariant to the iterations).

i	$n = 15$	$n = 30$	$n = 60$
0	0.41000	0.41000	0.41000
1	0.16132	0.13337	0.12205
2	0.07382	0.04560	0.03573
3	0.05370	0.02113	0.01266
4	0.05333	0.01628	0.00725
5	0.05323	0.01569	0.00659
6	0.05321	0.01556	0.00648
7	0.05320	0.01553	0.00646
8	0.05320	0.01552	0.00645
9	0.05319	0.01552	0.00645
...	...	...	...
∞	0.05319 (33)	0.01551 (30)	0.00645 (28)

Table 2: First order approximation: The width of $Y^{(i)}$ as a function of the iteration number i using $n = 15, 30, 60$ discretization intervals. The number in parenthesis is the total number of iterations before the algorithm stops (when $Y^{(i)}$ becomes invariant to the iterations).

Furthermore, since g has a singularity near $x = t = 0$, we use the non-equidistant subdivision $x_i = \frac{i^2}{n^2}$, $i = 0, ..., n$. The initial function interval $Y^{(0)}$ is $Y(x) \equiv [.99, 1.4]$. Table 1 shows the width of $Y^{(i)}$ as a function of the iteration number i using the zero

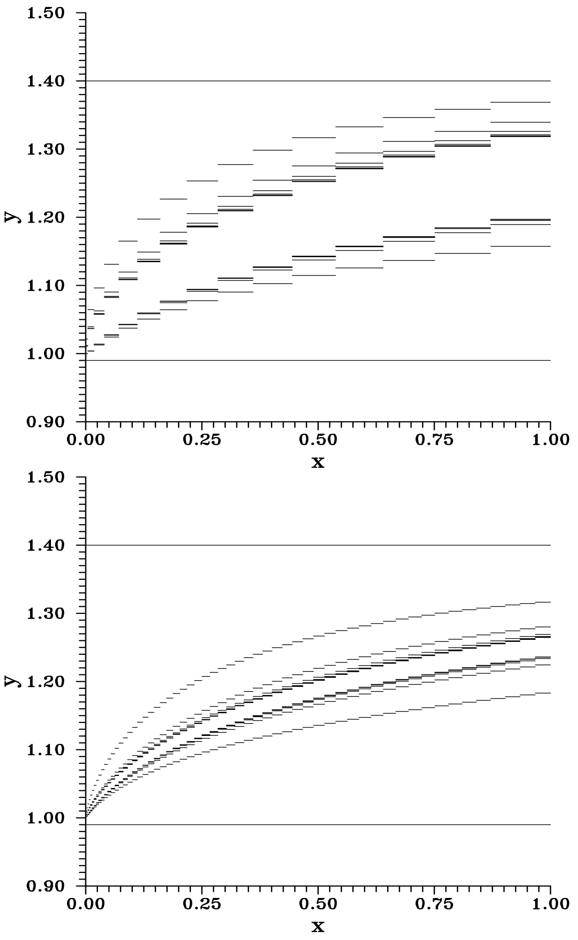

Figure 1: Zero order approximation: The iteration process solving the integral equation using $n = 15$ (top figure) and $n = 60$ (bottom figure) discretization intervals.

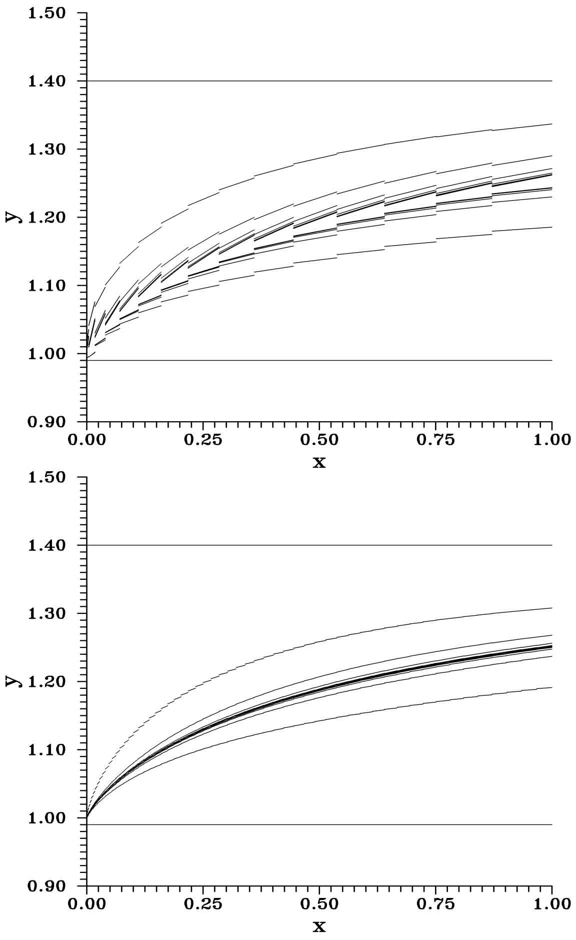

Figure 2: First order approximation: The iteration process solving the integral equation using $n = 15$ (top figure) and $n = 60$ (bottom figure) discretization intervals.

order approximation (12) with $n = 15, 30, 60$ discretization intervals. In Figure 1 the corresponding iteration processes are shown for $n = 15, 60$. Table 2 and Figure 2 show the results of the equivalent calculations using the first order approximation (14). All solutions were self-validated, i.e. the final function interval was proven to contain a solution to (15) because of the inclusion $IT_m Y^{(i)} \subseteq Y^{(i)}$.

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