Math. 301-01, Monday, 2007/03/19, Third exam answers ①

① (a) $\int_0^1 \frac{dx}{x^\pi}$ $\boxed{\text{diverges}}$ since $\pi > 1$: $\int_\varepsilon^1 \frac{dx}{x^\pi} = \frac{1}{1-\pi} x^{1-\pi}\Big|_\varepsilon^1 = \frac{1}{1-\pi}\left[1 - \frac{1}{\varepsilon^{\pi-1}}\right]$

$\to \infty$ as $\varepsilon \to 0$.

(b) $\int_1^\infty \frac{dx}{x^\pi}$ $\boxed{\text{converges}}$, since $\pi > 1$. In particular

$$\int_1^\infty \frac{dx}{x^\pi} = \lim_{M\to\infty} \frac{1}{1-\pi}\left[-1 + \frac{1}{M^{\pi-1}}\right] = \boxed{\frac{1}{\pi-1}}$$

(c) $\int_0^1 \frac{dx}{\pi^x}$ is an ordinary integral, since the integrand is continuous on an interval that contains both end points. Therefore, it $\boxed{\text{converges}}$.

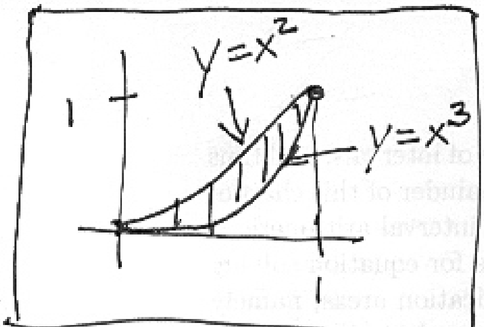
(d) $\int_1^\infty \frac{dx}{\pi^x} = \int_1^\infty \frac{dx}{e^{(\ln\pi)x}} = \int_1^\infty e^{-(\ln\pi)x}\,dx$

$$= \lim_{M\to\infty} -\frac{1}{\ln\pi} e^{-\ln\pi\, x}\Big|_{x=1}^{\infty} = \frac{1}{\ln\pi} e^{-\ln\pi} = \frac{1}{\pi\ln\pi} = \frac{1}{\ln(\pi^\pi)}.$$

Thus, the integral converges.

(e) ~~For $x \geq$, x^π~~ Basically, integrals of negative exponentials times powers of x converge. In particular, $x^\pi < x^4$ for $x \geq 1$, and $\int_1^\infty \frac{x^3}{\pi^x}dx$ converges by repeated integration by parts. Therefore, $\int_1^\infty \frac{x^\pi}{\pi^x}dx$ converges by the comparison test.

Math. 301-01, Monday, 2007/03/19, Third exam answers (2)



The total mass of the object is:

$$M=\int_{x=0}^{1}\rho\left[x^2-x^3\right]dx$$

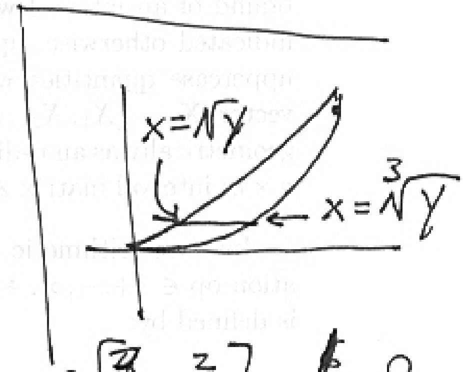
$$=\rho\left[\frac{x^3}{3}-\frac{x^4}{4}\right]\Big|_{x=0}^{1}=\rho\left[\frac{1}{3}-\frac{1}{4}\right]=\boxed{\frac{\rho}{12}}.$$

The x-moment is:

$$\int_{x=0}^{1}\rho\,x\left[x^2-x^3\right]dx=\rho\left[\frac{x^4}{4}-\frac{x^5}{5}\right]\Big|_{x=0}^{1}=\rho\left[\frac{1}{4}-\frac{1}{5}\right]=\boxed{\frac{\rho}{20}}$$

Thus, the x-coordinate of the center of mass is: $\cancel{\dfrac{\rho/12}{\rho/20}}=\dfrac{\rho/20}{\rho/12}=\dfrac{12}{20}=\boxed{\dfrac{3}{5}}=\bar{x}.$

Similarly, the y-moment is:



$$\int_{y=0}^{1}\rho\,y\left[y^{1/3}-y^{1/2}\right]dy$$

$$=\rho\int_{y=0}^{1}\left(y^{4/3}-y^{3/2}\right)dy=\rho\left[\frac{3}{7}y^{7/3}-\frac{2}{5}y^{5/2}\right]\Big|_{y=0}^{1}=\rho\left[\frac{3}{7}-\frac{2}{5}\right]=\frac{1}{35}\rho$$

Thus, $\bar{y}=\dfrac{1/35\,\rho}{1/12\,\rho}=\boxed{\dfrac{12}{35}}.$

Thus, the coordinates of the ~~solution~~ center of mass are:

$$\boxed{(\bar{x},\bar{y})=\left(\frac{3}{5},\frac{12}{35}\right)}$$