Math. 301-01
Fall, 2000
R. B. Kearfott

Final Examination

Friday, December 8, 2000, 7:30–10:00

Instructions: This exam should be done on your own paper. Your name should be on each sheet and on the back of the last sheet; the answers should appear written carefully and in order. If in doubt, show intermediate steps: Full credit may not be given, even for correct answers, unless work is arranged clearly and explained. This exam is closed book, but you may use your calculator and the computers. You may leave after handing in your exam paper, but be sure to check your answers carefully. Give exact values, rather than numerical approximations, unless the problem asks for a numerical approximation. Each entire problem is worth 11 points, and one point is "free."

1. If $F'(x) = f(x)$ is as in the graph in figure 1, and $F(0) = 0$, then draw a corresponding graph of $F(x)$. Be sure to label the axes on your graph.

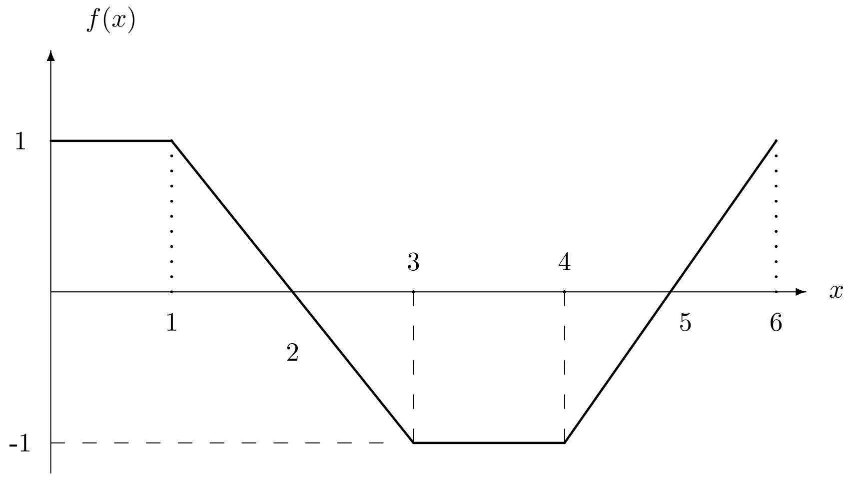

Figure 1: The derivative of the function you are to draw in Problem 1

2. Compute the following definite and indefinite integrals. Show all work, or state how you arrived at the answer; points will be deducted if you do not.

$$\text{(a)} \quad \int 2xe^{x^2}dx \qquad \text{(b)} \quad \int_{x=0}^{1} xe^{x}dx$$

3. State whether each of the following improper integrals converges. Supply values for those that do. (Any partial credit will be based on the work seen.)

$$\text{(a)} \quad \int_{x=0}^{\infty} xe^{-2x}dx \qquad \text{(b)} \quad \int_{x=0}^{1} \frac{dx}{x^2} \qquad \text{(c)} \quad \int_{x=1}^{\infty} \frac{dx}{x^2}$$

4. State whether each of these improper integrals converges. Explain why you have come to the conclusion you do. (Points will be deducted for failure to explain.)

$$\text{(a)} \quad \int_{x=0}^{\infty} (1+\cos x)e^{-x}dx \qquad\qquad \text{(b)} \quad \int_{x=0}^{\infty} \frac{e^x}{x^4}dx$$

5. Compute the volume of the solid obtained by rotating the curve $y = x^2$ for y between 1 and 4 about the y-axis.

6. In an electrical circuit, the resistance across two resistors connected in parallel is given by

$$R = \frac{R_1 R_2}{R_1 + R_2},$$

where the resistances of the two individual resistors are R_1 and R_2. Suppose R_1 is much larger than R_2.

   (a) Express R as R_2 times a function $f(R_2/R_1)$.

   (b) Write down the first three terms of the Taylor series for $f(x)$.

   (c) Suppose $R_1 = 1000$ ohms and $R_2 = 3$ ohms. Compute an approximation to R by taking terms of degree 0 and 1 to the Taylor series for $f(R_2/R_1)$ expanded about $x = R_2/R_1 = 0$. Compare that to the "exact" value for R obtained by calculating directly from the formula. (Note: It won't be exact, since there will be roundoff error when you use your calculator, but there won't be an approximation error.)

7. Compute a bound on the error if $f(x) = \sin 2x$ is approximated by a Taylor polynomial of degree 3 (or 4) about $x = 0$ over the interval $x \in [-0.1, 0.1]$. Show all work.

8. Find the solution to the initial value problem

$$\frac{dP}{dt} = 0.02P, \quad P(0) = 200.$$

9. The radioactive isotope carbon-14 is present in small quantities in all life forms, and is constantly replenished until the organism dies, after which it decays to stable carbon-12 at a rate proportional to the amount of carbon-14 present, with a half-life of 5730 years. Suppose $C(t)$ is the amount of carbon-14 present at time t.

   (a) Find the value of the constant k in the differential equation $C' = -kC$.

   (b) In 1988 three teams of scientists found that the Shroud of Turin, which was reputed to be the burial cloth of Jesus, contained 91% of the amount of carbon-14 contained in freshly made cloth of the same material. How old is the Shroud of Turin, according to this data?