(1)

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\sin(x)$	1
1	$\cos(x)$	0
2	$-\sin(x)$	-1
3	$-\cos(x)$	0
4	$\sin(x)$	1
5	$\cos(x)$	0
6	$-\sin(x)$	-1

The degree n term is of the form $f^{(n)}(\pi/2)\dfrac{(x-\pi/2)^n}{n!}$. Filling into the table, we thus have:

$$\boxed{P_6(x) = 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} - \frac{(x-\pi/2)^6}{6!}}$$

Note: If the Taylor polynomial for $\cos(x)$ at $x=0$ is known, we may obtain the desired polynomial by observing $\cos(x) = \sin(x+\pi/2)$.

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(2) ~~If~~ If $g(u) = \frac{1}{1+u} = (1+u)^{-1}$, we may use the binomial expansion to obtain

$$g(u) \approx 1 + (-1)u + \frac{(-1)(-2)}{2}u^2 + \frac{(-1)(-2)(-3)}{3!}u^3$$
$$\approx 1 - u + u^2 - u^3.$$

Plugging in $u = x^2$, we obtain:

$$\boxed{f(x) \approx 1 - x^2 + x^4 - x^6}$$

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(3) (a) The Taylor series for f is

$$\frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots}{x} - 1 = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} \cdots - 1 = \frac{-x^2}{3!} + \frac{x^4}{5!} + \cdots$$

(b) The Taylor series for g is:

$$\frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots - 1}{x} - 1 = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots -$$

$$= \frac{x}{2!} + \frac{x^2}{3!} + \cdots$$

(c) The Taylor series for $\dfrac{\log(1+x)-x}{x^2}+\dfrac{1}{2}$

is: $\dfrac{x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots-x}{x^2}+\dfrac{1}{2}=\dfrac{-\frac{x^2}{2}+\frac{x^3}{3}+\cdots}{x^2}+1/2$

$=\frac{-1}{2}+\frac{x}{3}-\frac{x^2}{4}+\cdots+\frac{1}{2}=\frac{x}{3}-\frac{x^2}{4}+\cdots$

Comparing: $f(x)=-x^2/3!+\cdots$
$g(x)=x/2!+\cdots$
$h(x)=\frac{x}{3}+\cdots$

We see, for $|x|$ small, $x>0$, $f<0$, $g\approx\frac{x}{2}$, $h\approx\frac{x}{3}$.

Thus, for $x>0$ and $|x|$ sufficiently small, $f(x)\le h(x)\le g(x)$.

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(4) First note $|\cos(x)|\le 1$, and $|\sin(x)|\le 0.1$ for $|x|\le 0.1$.
~~If we evaluate the su~~ Also note that only even order terms occur in the taylor polynomial, so the error term is going to be of the form

$$\left|\cancel{\cos}^{(2n+1)}(c)\right|\frac{|x|^{2n+1}}{(2n+1)!}=|\sin(c)|\frac{|x|^{2n+1}}{(2n+1)!}\le\frac{(0.1)(0.1)^{2n+1}}{(2n+1)!}=\frac{(0.1)^{2n+2}}{(2n+1)!}$$

We form a table:

$2n$	Error bound
2	$(0.1)^4/3! \approx 1.\bar{6}\times10^{-5}$
4	$(0.1)^6/5! \approx 8.3\times10^{-9}$
6	~~8.3×10~~

Thus, we may take

$$\boxed{P_4(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}}.$$