Math. 301-03, Final exam answers, Monday, 12/5/2011, page 1

(a) $$\frac{d}{dx}\int_0^x \ln(\cos(e^{-\sin(t)}))\,dt = \boxed{\ln(\cos(e^{-\sin(x)}))}.$$

(b) $$\int \frac{dx}{x^2+3x+2} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2}$$

$$= \ln|x+1| - \ln|x+2| + C$$

$$= \cancel{\ln|x^2+3x} \boxed{\ln\left|\frac{x+1}{x+2}\right| + C}.$$

$$\frac{1}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$1 = A(x+2)+B(x+1)$$
$$= (A+B)(x) + (2A+B)$$
$$A+B=0 \Rightarrow B=-A$$
$$\boxed{A=1,\ B=-1}$$

(c) $$\int_{x=0}^{\sqrt[3]{\pi}} x^2\sin(x^3)\,dx = \int_{u=0}^{\pi} \sin(u)\,\frac{du}{3}$$

$u = x^3$

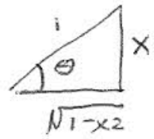
$\frac{du}{3} = x^2 dx$

$$= \frac{1}{3}\left[-\cos u\right]\Big|_{u=0}^{\pi} = -\frac{1}{3}\left[-1-1\right] = \boxed{\frac{2}{3}}$$

(d) $$\int_{-\pi}^{\pi} x\sin x\,dx = -x\cos x\Big|_{x=-\pi}^{\pi} + \int_{-\pi}^{\pi}\cos(x)\,dx$$

$$= -\pi\cos(\pi) + (-\pi)\cos(-\pi) = \boxed{2\pi}$$

(e) $$\int_0^{\sqrt{2}/2} \frac{dx}{\sqrt{1-x^2}} = \int_{\theta=0}^{\pi/4} \frac{\cos(\theta)\,d\theta}{\cos(\theta)} = \int_0^{\pi/4} d\theta = \boxed{\frac{\pi}{4}}$$




$x = \sin(\theta)$

$dx = \cos(\theta)\,d\theta$

$\sqrt{1-x^2} = \cos(\theta)$

(f) $$\int_{x=0}^{\infty} x e^{-x^2}\,dx = \int_{u=0}^{\infty} e^{-u}\,\frac{du}{2}$$

$u = x^2$

$\frac{du}{2} = x\,dx$

$$= \lim_{m\to\infty}\left[-\frac{1}{2}e^{-u}\right]\Big|_{u=0}^{m}$$

$$= \lim_{m\to\infty}\left[\frac{1}{2} - \frac{1}{2}e^{-m}\right] = \boxed{\frac{1}{2}}.$$

1. $\displaystyle\int_{-1}^{1}\frac{dx}{\sqrt{1-x}} = \int_{-1}^{1}(1-x)^{-1/2}dx = -2(1-x)^{1/2}\Big|_{x=-1}^{1} = \boxed{2\sqrt{2}}$.

2. $\dfrac{dv}{dt} = a \Rightarrow v(t) = at + v_0 = at$ since $v_0 = 0$.

$\dfrac{ds}{dt} = v = at \Rightarrow s(t) = \dfrac{at^2}{2} + s_0$. $s_0 = 0 \Rightarrow s(t) = a\dfrac{t^2}{2}$

(a) $s(0.25) = 4 = a(0.25)^2/2 = \dfrac{a}{32} \Rightarrow \boxed{a = 128 \text{ miles per hour per hour}}$.

(b) $v(0.25) = \dfrac{128}{4} = \boxed{32 \text{ miles per hour}}$.

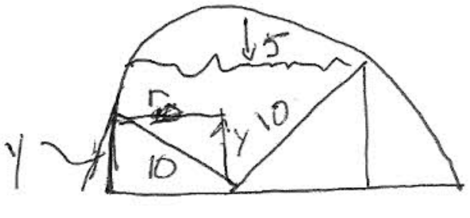
3. (a) $\left|\dfrac{\sin^2(x)}{x^2}\right| \le \dfrac{1}{x^2}$ and $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges, so

$\displaystyle\int_1^\infty \frac{\sin^2(x)}{x^2}$ converges

(b) $\displaystyle\int_0^1 \frac{dx}{x^2} = \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{dx}{x^2} = \lim_{\varepsilon\to 0^+} \frac{-1}{x}\Big|_{x=\varepsilon}^{1} = \lim_{\varepsilon\to 0^+} -1+\frac{1}{\varepsilon}$

does not exist: This integral diverges.

4. This was problem 1 in Exam 3b:




$r = \sqrt{100-y^2}$


[Volume] $= \pi r^2 \Delta y$

The total volume inside the tank filled to $10-5=5$ ft. deep is

$$\int_{y=0}^{5}\pi\left(\sqrt{10^2-y^2}\right)^2 dy = \pi\int_0^5 (100-y^2)\,dy$$

$$= \pi\left(100y - \frac{y^3}{3}\right)\Big|_{y=0}^{5} = \pi\left(500 - \frac{125}{3}\right) \approx 1439.9 \text{ cubit feet.}$$

5 This was problem 2 in Exam 3:

The total length of cable is about

$$\int_0^1 \sqrt{1+(y'(x))^2}\,dx = \int_0^1 \sqrt{1+\sinh^2(x-0.5)}\,dx = \int_0^1 \sqrt{\cosh^2(x-0.5)}\,dx$$

$$= \int_0^1 \cosh(x-0.5)\,dx = \sinh(x-0.5)\Big|_{x=0}^{1} = \sinh(0.5)-\sinh(-0.5)$$

$$= 2\sinh(0.5) \boxed{\approx 1.042 \text{ units}}.$$

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6 $\sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots$

$\boxed{P_3(0.1) \approx 0.0999\overline{4}}$

so $\frac{\sin(t)}{t} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \cdots$

so $\int_{t=0}^{x} \frac{\sin(t)}{t}\,dt = \int_0^x 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \cdots dt = \boxed{x - \frac{x^3}{3(3!)}} + \frac{x^5}{5(5!)} - \cdots$

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7 a $C_n = \frac{(-1)^{n-1}2^n}{n}$. The radius of convergence is

$$\lim_{n\to\infty}\left|\frac{C_n}{C_{n+1}}\right| = \lim_{n\to\infty}\left(\frac{n+1}{2n}\right) = \boxed{\frac{1}{2}}.$$

The interval of convergence is thus

$-\frac{1}{2} < x-1 < \frac{1}{2}$, that is $\frac{1}{2} < x < 3/2$.

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b Checking the end points:

$\underline{x=\frac{1}{2}}$: $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n}{n}\left(\frac{-1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n}$

$= \sum_{n=1}^{\infty} -\frac{1}{n}$ diverges, since it is a harmonic series.

$x = 3/2$: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n}{n}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

converges by the alternating series test.