Math. 301-02, July 13, 2005, Third Exam answers, page 1

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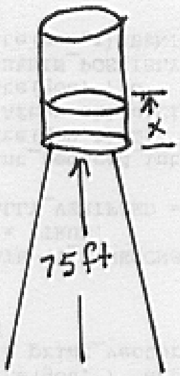

A disk-shaped layer of water at height x from the bottom of the tank has cross-sectional area 100π, volume $100\pi\Delta x$, and total weight $(100\pi\Delta x)(62.4) = 6240\pi\Delta x$.

The work required to lift this weight from the ground is thus $(75+x)(6240\pi\Delta x)$ (approximately). Summing over all such layers of water and taking the limit as $\Delta x \to 0$ thus gives:

$$[\text{Total work}] = \int_{x=0}^{30} (75+x)(6240\pi\, dx)$$

$$= 6240\pi \int_{x=0}^{30} (75+x)\,dx = 6240\pi\left[75x + \frac{x^2}{2}\right]\Bigg|_{x=0}^{30}$$

$$= 6240\pi\left[75(30) + \frac{900}{2}\right] \approx \cancel{59}\ 52{,}929{,}553 \text{ ft.lb.}$$

2 (a) The equilibrium quantity is where $C(q) = p(q)$

This gives $1500 - 0.1\sqrt{q^*} = 220 + 0.2\sqrt{q^*}$,

whence $1280 = 0.2\sqrt{q^*} \Rightarrow q^* = \left(\frac{1280}{0.2}\right)^2 = 40{,}960{,}000.$

The equilibrium price is: $p(q^*) = 860 = p^*$

(b) The consumer surplus is:

$$\int_{q=0}^{40{,}960{,}000} (1500 - 0.1q^{1/2})\,dq - (40{,}960{,}000)(860)$$

$$= \left[1500q - \frac{2}{3}(0.1)q^{3/2}\right]\Big|_{q=0}^{40{,}960{,}000} - (40{,}960{,}000)(860)$$

$\approx 8{,}738{,}133{,}333$ (about 9 billion).

The producer surplus is:

$$(40{,}960{,}000)(860) - \int_{q=0}^{40{,}960{,}000} (220 + 0.1q^{1/2})\,dq$$

$$= (40{,}960{,}000)(860) - \left[220q + (0.1)\left(\frac{2}{3}\right)q^{3/2}\right]\Big|_{q=0}^{40{,}960{,}000}$$

$\approx 8{,}738{,}133{,}333$ (the same as the consumer surplus, in this case).

3. The cumulative probability is:

(a)
$$P(x) = \int_{s=0}^{x} \frac{1}{4} e^{-\frac{1}{4}s}\,ds = -e^{-\frac{1}{4}s}\Big|_{s=0}^{x} = \boxed{1-e^{-.25x} = P(x)}$$

(b)
$$P(1 \le x \le 2) = \int_{s=1}^{2} \frac{1}{4} e^{-.25s}\,ds = -e^{-1/4 s}\Big|_{s=1}^{2}$$
$$= e^{-1/4} - e^{-1/2} \approx .172$$

(c)
$$\bar{x} = \int_{s=0}^{\infty} \frac{1}{4} s\, e^{-\frac{1}{4}s}\,ds = -s e^{-\frac{1}{4}s}\Big|_{s=0}^{\infty} + \int_{s=0}^{\infty} e^{-\frac{1}{4}s}\,ds$$
$$= -4 e^{-1/4 s}\Big|_{s=0}^{\infty} = \boxed{4}.$$

(d) The median is where $P(x) = \frac{1}{2}$. This gives
$$1 - e^{-.25x} = \frac{1}{2} \Rightarrow e^{-.25x} = \frac{1}{2} \Rightarrow -.25x = \ln(1/2)$$
$$= x = -4\ln\left(\frac{1}{2}\right) = 4\ln(2) \approx \boxed{2.77}$$

(4) (a) $100 + 100(1.0025) + \cdots + 100(1.0025)^n$

$$= \sum_{i=0}^{n} 100(1.0025)^i = S_n$$

(b) $$S_n = 100\left[\frac{(1.0025)^{n+1} - 1}{1.0025 - 1}\right].$$

(c) 5 years corresponds to 60 months, so, after the end of 60 months, you would have

$$(1.0025)\,S_{59} = (1.0025)\left[\frac{(1.0025)^{60} - 1}{.0025}\right]100$$

$$\approx \$6,480.83$$

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(5) $\sum_{i=0}^{\infty} \frac{(-1)^i}{i!}$ converges, since it is an alternating series whose terms have absolute values that are decreasing to 0. (In fact, it can be shown that the series converges absolutely by the ratio test: $\left|\frac{a_{i+1}}{a_i}\right| = \frac{1}{i+1} \to 0 < 1$ as $i \to \infty$.)

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a) $$r = \lim_{i \to \infty} \frac{\frac{1}{i!}}{\frac{1}{(i+1)!}} = \lim_{i \to \infty} (i+1) = \infty.$$ Thus, the series converges for all x.

b) $$r = \lim_{i \to \infty} \frac{\frac{1}{i}}{\frac{1}{i+1}} = \lim_{i \to \infty} \frac{i+1}{i} = 1.$$ Thus, the series converges for $|x| < 1$.