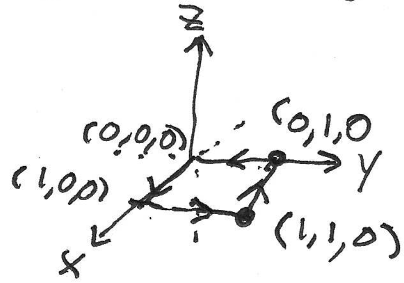
Math. 302-05, Fall, 2016, Fifth exam answers, page 1

(1) $\oint_{C} \vec{F}\circ d\vec{r} = \iint_{\sigma} \operatorname{curl}(\vec{F})\circ d\vec{s}$



$\vec{s}(u,v) = \langle u, v, 0\rangle$, $0\le u\le 1$, $0\le v\le 1$,
with unit normal $\langle 0,0,1\rangle$.

$$d\vec{s} = \frac{\partial \vec{s}}{\partial u}\times\frac{\partial \vec{s}}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \vec{k}, \text{ while}$$

$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & z \end{vmatrix} = 0\vec{i} + 0\vec{j} + 2\vec{k}.$$

Therefore, $\iint_{\sigma} \operatorname{curl}(\vec{F})\circ d\vec{s} = \int_{u=0}^{1}\int_{v=0}^{1} \langle 0,0,2\rangle\circ\langle 0,0,1\rangle \, dv\, du$

$$= \int_{u=0}^{1}\int_{v=0}^{1} 2\, dv\, du = \boxed{2}.$$

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(2) $\iint_{\sigma} \vec{F}\circ d\vec{s} = \iiint_{V} \operatorname{div}(\vec{F})\, dV.$

$\operatorname{div}(\vec{F}) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 0 + 0 + 1 = 1$, so the flux

out is $\iint_{\sigma} \vec{F}\circ d\vec{s} = \iiint_{V} 1\, dV = 1\cdot$[volume of the unit sphere]

$= \boxed{\frac{4}{3}\pi}$

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(3) $dS = \left\|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right\| du\,dv = \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 0 & 0 & 2v \end{vmatrix} \right\| du\,dv$

$= \|2v(\vec{i}-\vec{j})\|\,du\,dv = 2\sqrt{2}\,v\,du\,dv$. Furthermore,

$f(x,y,z) = u+v^2$; so

$$\iint_{S} f(x,y,z)\,dS = \int_{u=0}^{1}\int_{v=0}^{2} (u+v^2)2\sqrt{2}\,v\,dv\,du$$

$$= \int_{u=0}^{1}\int_{v=0}^{2} 2\sqrt{2}(uv+v^3)\,dv\,du = 2\sqrt{2}\int_{u=0}^{1}\left[\frac{uv^2}{2}+\frac{v^4}{4}\right]\Big|_{v=0}^{2} du$$

$$= 2\sqrt{2}\int_{u=0}^{1}(2u+4)\,du = 2\sqrt{2}\left[u^2+4u\right]\Big|_{u=0}^{1}$$

$$= 2\sqrt{2}[1+4] = \boxed{10\sqrt{2}}$$