MATH 304–02: CALCULUS III – DR. KEARFOTT
TEST 1 SOLUTIONS

Problem 1. Write down an equation for the sphere with center $(1, -1, 3)$ and radius 2.

Solution. In general the standard form equation for a sphere with radius r and center (x_0, y_0, z_0) is given by: $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$. Thus for this sphere our equation is:

$$(x - 1)^2 + (y + 1)^2 + (z - 3)^2 = 4.$$

Problem 2. Describe as precisely as possible the surface given by the equation $x^2 + y^2 + z^2 + 6x + 8y = 0$.

Solution. First we want to get this equation into standard form by completing the square:

$$\begin{aligned} x^2 + y^2 + z^2 + 6x + 8y = 0 &\iff (x^2 + 6x) + (y^2 + 8y) + z^2 = 0 \\ &\iff (x + 3)^2 + (y + 4)^2 + z^2 = 0 + 9 + 16 \\ &\iff (x + 3)^2 + (y + 4)^2 + z^2 = 25. \end{aligned}$$

It follows that we have a sphere of radius 5 centered at $(-3, -4, 0)$.

Problem 3. Find

(a) the projection of $\langle 3, 4, 6\rangle$ onto $\langle 1, 1, 0\rangle$ and

Solution. Note that

$$\begin{aligned} \text{proj}_{\langle 1,1,0\rangle}(\langle 3, 4, 6\rangle) &= \left(\frac{\langle 3, 4, 6\rangle \cdot \langle 1, 1, 0\rangle}{\langle 1, 1, 0\rangle \cdot \langle 1, 1, 0\rangle}\right) \langle 1, 1, 0\rangle \\ &= \left\langle \frac{7}{2}, \frac{7}{2}, 0 \right\rangle. \end{aligned}$$

(b) find the length of that projection.

Solution. Note that

$$\begin{aligned} \left\| \left\langle \frac{7}{2}, \frac{7}{2}, 0 \right\rangle \right\| &= \sqrt{\left(\frac{7}{2}\right)^2 + \left(\frac{7}{2}\right)^2} \\ &= \frac{7\sqrt{2}}{2}. \end{aligned}$$

Problem 4. Consider the points $(1,0,0),(0,1,0)$ and $(0,0,1)$

(a) Use the cross product to find the area of the triangle with these points as verticies.

Solution. Note that

$$\begin{aligned}
\boldsymbol{P_1P_2} &= \langle 0,1,0\rangle - \langle 1,0,0\rangle = \langle -1,1,0\rangle \\
\boldsymbol{P_1P_3} &= \langle 0,0,1\rangle - \langle 1,0,0\rangle = \langle -1,0,1\rangle \\
\text{Area} &= \frac{1}{2}\,||\boldsymbol{P_1P2} \times \boldsymbol{P_1P_3}|| \\
&= \frac{1}{2}\left|\left|\det\left(\begin{bmatrix} i & j & k \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\right)\right|\right| \\
&= \frac{1}{2}\,||\langle 1,1,1\rangle|| \\
&= \frac{\sqrt{3}}{2}.
\end{aligned}$$

(b) Find a unit normal vector to the plane in which the triangle lies.

Solution. Observe that $\langle 1,1,1\rangle$ is perpendicular to both $\boldsymbol{P_1P_2}, \boldsymbol{P_1P_3}$, both of which line in the plane in question. It follows that

$$\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

is a unit normal vector.

Problem 5. Show that the following lines intersect, and find their point of intersection.

$$\begin{aligned}
x &= t, & y &= t, & z &= t, \\
x &= 2u-1, & y &= -u, & z &= -u.
\end{aligned}$$

Solution. We want the lines to intersect. It this is the case then

$$\begin{aligned}
&\Rightarrow t = 2u-1, t = -u \\
&\Rightarrow 2u-1 = -u \\
&\Rightarrow 3u = 1 \\
&\Rightarrow u = \frac{1}{3} \\
&\Rightarrow t = -\frac{1}{3}.
\end{aligned}$$

Thus our intersection (we have an intersection since we arrived at no contradictions) is the point

$$\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right).$$

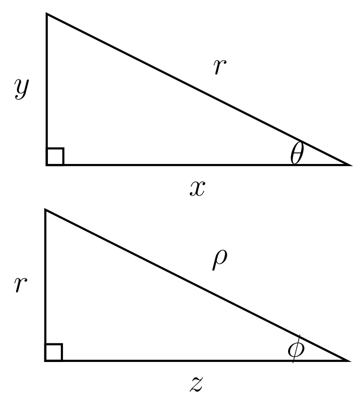
Problem 6. Determine an equation for the plane perpendicular to the line through $(0,0,0)$ and $(1,1,1)$ and containing the point $(2,3,4)$

Solution. We want the norm of our plane to be parallel to $\langle 1,1,1\rangle$ (this implies that the plane is perpendicular to $\langle 1,1,1\rangle$). And we want to pass through $(2,3,4)$. This gives

$$1(x-2)+1(y-3)+1(z-4)=0$$
$$\iff$$
$$x+y+z=9.$$

Problem 7. Consider the spherical coordinate equation $\rho\sin\phi = 1$.

(a) Write down a corresponding equation in rectangular coordinates. *Solution.* From the two triangles:



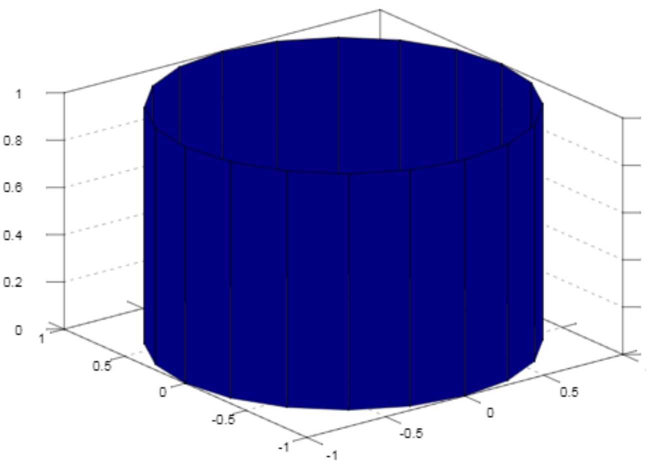
We get that:

$$\rho\sin\phi = \left(\sqrt{x^2+y^2+z^2}\right)\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$
$$= \sqrt{x^2+y^2} = 1$$
$$\Rightarrow x^2+y^2 = 1$$

is the given equation in rectangular coordinates.

(b) Precisely describe the graph of this equation.

Solution. This is a cylinder of radius one, centered at the origin, and with extrusion in



the z direction.