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(1) $$\iint_R x^2+y^2\,dA = \int_{y=0}^{2}\int_{x=0}^{1}(x^2+y^2)\,dx\,dy = \int_{y=0}^{2}\left[\frac{x^3}{3}+xy^2\right]_{x=0}^{1}dy$$

$$= \int_{y=0}^{2}\frac{1}{3}+\cancel{\tfrac{1}{3}}y^2\,dy = \left[\frac{1}{3}y+\frac{1}{3}y^3\right]\Big/_{y=0}^{2} = \frac{2}{3}+\frac{8}{3} = \boxed{\frac{10}{3}}$$

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(2) $$\iint_D x^2+y^2\,dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{\left(\frac{20}{3\pi}\right)^{1/4}} r^2(r\,dr)\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\left[\frac{r^4}{4}\right]\Big/_{r=0}^{\left(\frac{20}{3\pi}\right)^{1/4}}d\theta = \int_{\theta=0}^{2\pi}\frac{20}{12\pi}\,d\theta = 2\pi\left(\frac{20}{12\pi}\right) = \boxed{\frac{10}{3}}$$

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(3) Tangents to the surface are $\langle 1,0,x\rangle$ and $\langle 0,1,y\rangle$,
~~so~~ with cross product $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & x \\ 0 & 1 & y \end{vmatrix} = \vec{i}(-x)-\vec{j}(y)+\vec{k}(1)$
$= \langle -x,-y,1\rangle.$

Thus, the required surface area is:

$$\iint_D \|\langle -x,-y,1\rangle\|\,dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\sqrt{x^2+y^2+1}\;r\,dr\,d\theta.$$

With $u=r^2+1$, we have $\frac{du}{2}=dr$, and the integral becomes

$$\frac{1}{2}\int_{\theta=0}^{2\pi}\int_{u=1}^{2}u^{1/2}\,du\,d\theta = \frac{2\pi}{3}u^{3/2}\Big/_{u=1}^{2} = \boxed{\frac{2\pi}{3}(2\sqrt{2}-1)}$$

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(4) $$\iiint_{\mathcal{V}} \frac{1}{\sqrt{x^2+y^2+z^2}}\,dV = \int_{\rho=0}^{1}\int_{\varphi=0}^{\pi}\int_{\theta=0}^{2\pi} \frac{1}{\rho}(\rho^2 \sin\varphi)\,d\theta\,d\varphi\,d\rho$$

$$= \left\{\int_{\rho=0}^{1} \rho\,d\rho\right\}\left\{\int_{\varphi=0}^{\pi} \sin\varphi\,d\varphi\right\}\left\{\int_{\theta=0}^{2\pi} d\theta\right\} = \left(\tfrac{1}{2}\right)(2)(2\pi) = \boxed{2\pi}.$$