Math. 302-02, Summer, 2015, Final exam answers, page ①

① We use spherical coordinates: $dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$.

$(x^2+y^2+z^2)^{1/2} = \rho \qquad 0 \le \rho \le 2, \; 0 \le \theta \le 2\pi, \; 0 \le \varphi \le \pi/2$, so

$$\iiint_{Q} e^{(x^2+y^2+z^2)^{3/2}} \, dV = \int_{\rho=0}^{2} \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/2} e^{\rho^3} (\rho^2 \sin\varphi) \, d\varphi \, d\theta \, d\rho$$

$$= \left(\int_{\rho=0}^{2} \rho^2 e^{\rho^3} d\rho \right) \left(\int_{\theta=0}^{2\pi} d\theta \right) \left(\int_{\varphi=0}^{\pi/2} \sin\varphi \, d\varphi \right)$$

$$\left. \begin{matrix} u = \rho^3 \\ du = 3\rho^2 d\rho \end{matrix} \right\} = \left(\frac{1}{3} \int_{u=0}^{8} e^u \, du \right) (2\pi)(1) = \boxed{\frac{2\pi}{3}(e^8 - 1)} \approx 6.24 \times 10^3$$

② The line is parametrized by $\vec{r}(t) = (1+t, 1+2t, 1+3t)$.

Plugging these coordinates into the plane gives.

$$(1+t) + (1+2t) + (1+3t) = 1$$

$$3 + 6t = 1 \Rightarrow 6t = -2, \; \boxed{t = -1/3}.$$

This gives $\boxed{(x, y, z) = \left(\frac{2}{3}, \frac{1}{3}, 0\right)}$ is the point of intersection.

③ $\iint_S \vec{E} \circ d\vec{A}$ is to be computed, where $d\vec{A} = (3\cos\theta, 3\sin 3, 0) \, d\theta \, dz$;

$x = 3\cos\theta, \; y = 3\sin\theta, \; z = z, \qquad 0 \le \theta \le 2\pi, \; -1 \le z \le 1$.

This gives $\iint_S \vec{E} \circ d\vec{A} = \int_{z=-1}^{1} \int_{\theta=0}^{2\pi} (3\cos\theta, 3\sin\theta, \dots) \circ (3\cos\theta, 3\sin\theta, 0) \, d\theta \, dz$

$$= 9 \int_{z=-1}^{1} \int_{\theta=0}^{2\pi} \cos^2(\theta) + \sin^2(\theta) \, d\theta \, dz = 9(2)(2\pi) = \boxed{36\pi}$$

(4) We will use the divergence theorem, if possible.

$$\iint_S \vec{F}\circ d\vec{A} = \iiint_V \operatorname{div}(\vec{F})\,dV$$

Now $\operatorname{div}(\vec{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 1 + \cancel{2x} - 2y + 1 = 2$,

so $\iiint_V \operatorname{div}(\vec{F})\,dV = \iiint_V 2\,dV = 2(5) = \boxed{10}$.

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(5) We'll see if we can compute a potential function.

$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \vec{i}\,(x - x) - \vec{j}\,(y-y) + \vec{k}\,(z-z) = \vec{0},$$

so $\vec{F}$ is a gradient field. If f is a potential function,

$f_x = yz \Rightarrow f = xyz + g(y,z)$,

so $f_y = xz + \frac{\partial g}{\partial y} = xz$, so $g(y,z) = h(z)$,

so $f = xyz + h(z)$, so $f_z = xy + h'(z) = xy \Rightarrow h(z) = C$.

Thus $f = xyz + C$.

Therefore, $\int_C \vec{F}\circ d\vec{r} = f(1,1,1) - f(0,0,0) = 1 - 0 = \boxed{1}$

6 a) We parametrize the circle with $\left.\begin{array}{l} x=2 \\ y=\cos(t) \\ z=\sin(t) \end{array}\right\} 0 \le t \le 2\pi$,

$d\vec{r} = (0, -\sin t, \cos(t))dt$, $\vec{F} = (2, 2, \sin(t))$,

so $\int_{C} \vec{F} \circ d\vec{r} = \int_{t=0}^{2\pi} (-2\sin t + \sin(t)\cos(t)) = \boxed{0}$.

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b) Using Stokes' theorem $\int_{C} \vec{F} \circ d\vec{r} = \iint_{S} \operatorname{curl}(\vec{F}) \circ d\vec{A}$,

where S is any surface that C bounds.

$\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & x & z \end{vmatrix} = \vec{i}(0) - \vec{j}(0) + \vec{k}(1) = (0,0,1)$

However, $d\vec{A}$ points in the direction of $(1,0,0)$,

so $\operatorname{curl}(\vec{F}) \circ d\vec{A} = 0$, so $\iint_{S} \operatorname{curl}(\vec{F}) \circ d\vec{A} = \boxed{0}$