Math. 350-01
Fall, 2006
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First Exam

Wednesday, September 13

This exam is closed book. Make sure your name is on all pages. Show all work, and show it in a logical and organized manner. You may keep this exam sheet.

1. (20 points) State whether each of the following differential equations is linear or nonlinear. Explain why, as appropriate. Also state the order of each equation, and state what the independent variable is and what the dependent variable is.

   (a) $y' + ty = t^2$ (b) $y'' + \sin(t)y' + \cos(t)y = e^t$
   (c) $y'' + yy' + y = 0$ (d) $y'' + \sin(y) = \cos(t)$

2. (15 points) Use pencil and paper to derive the solution to the initial value problem

$$\frac{dy}{dt} + 2ty = 2t, \quad y(0) = 2.$$

3. (15 points) Use pencil and paper to find the general solution to the following differential equation. Describe the solution curves.

$$y'(x) = -\frac{y}{x}$$

4. (25 points) Carbon 14 is a radioactive isotope of carbon that is produced by cosmic rays in the atmosphere, then absorbed by living things, such as wood, bones, etc. After those things die, the carbon 14 decays, with a half-life of 5,730 years.

   (a) Suppose the initial amount of carbon 14 in a piece of linen cloth is 100 units.
      i. Write down a differential equation relating the rate of change of carbon 14 to the constant α of proportionality of the instantaneous rate of decay to the amount present.
      ii. Solve that differential equation in terms of α.
      iii. Use the half-life to determine the rate constant α.
      iv. Solve the initial value problem to determine the amount $y(t)$ that is present t years after the cloth was produced.
   (b) Suppose that roughly 94% of the original carbon 14 remains in the cloth. Approximately how old is the cloth?

5. (25 points) A second-order chemical reaction involves combining one molecule of substance P with one molecule of substance Q to produce one molecule of substance X; chemists denote this process by $P + Q \rightarrow X$. If p is the initial amount of substance P, q is the initial amount of substance Q, and $x = x(t)$ is the amount of the product X at time t, then the amount of substance P at time t is $p - x$, while the amount of substance Q at time t is $q - x$. Furthermore, the rate of production of substance X is proportional to the amount of substance P times the amount of substance Q, that is

$$x' = \alpha(p - x)(q - x)$$

   for some constant of proportionality α.

   (a) Assuming $x(0) = 0$ and $p = q$, find an expression for $x(t)$.
   (b) What is the final value of x as $t \rightarrow \infty$?