Math. 350-01, Fall, 2006, Second Exam answers 10/4/2006 ①

① (a) $y_0 = 1$, $y_1 = 1 - .1(1)^2 = .9$; $y_2 = .9 - .1(.9)^2 = .9 - .081 = .819$

(b) $y_0 = 1$. $y_1 = 1 - (.05)(1)^2 = .95$.

$y_2 = .95 - .05(.95)^2 \approx .904875$

$y_3 = y_2 - .05(y_2^2) \approx .8639$; $y_4 \approx .8639 - .05(.8639)^2 \approx .8266$

(c) $\int \frac{dy}{y^2} = \int -dt \Rightarrow \frac{-1}{y} = -t + \tilde{c} \Rightarrow \frac{1}{y} = t + C \Rightarrow y = \frac{1}{t+C}$

$y(0) = 1 = \frac{1}{C} \Rightarrow C = 1 \Rightarrow y(t) = \frac{1}{t+1}$.

$y(.2) = \frac{1}{1.2} \approx .8333$. The second answer, from part b, is closer than the answer from part ⓐ.

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② $y'' - y' - 2y = 0$ has characteristic equation $\alpha^2 - \alpha - 2 = 0$. This factors to $(\alpha - 2)(\alpha + 1) = 0$, so the solutions are $\alpha = 2$ and $\alpha = -1$. This gives $\boxed{y(t) = C_1 e^{2t} + C_2 e^{-t}}$.

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③ $y'' + y' + y = 0$ has char. eq. $\alpha^2 + \alpha + 1 = 0$.

Using the quadratic formula: $\alpha = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$.

Thus, the general solution is:

$y_g(t) = C_1 e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right)$.

$y(0) = C_1 = 0$, $y'(t) = \frac{-1}{2} C_2 e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{2} C_2 e^{-\frac{1}{2}t} \cos \frac{\sqrt{3}}{2}t$

so $y'(0) = \frac{\sqrt{3}}{2} C_2 = 1 \Rightarrow C_2 = 2/\sqrt{3}$, so

$$\boxed{y(t) = \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right)}.$$

(4) $y''+6y'+9y=0$ has a double root $y=-3$ to its characteristic equation. Therefore, the general solution is:

$$y_g(t) = C_1 e^{-3t} + C_2 t e^{-3t}.$$

$y(0) = C_1 = 0$, so $y'(t) = C_2 \left(e^{-3t} - 3te^{-3t}\right)$,
so $y'(0) = C_2 = 1$, so $\boxed{y(t) = te^{-3t}}$.

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(5) The general solution is a sum of the solution to the homogeneous equation plus any particular solution to the nonhomogeneous equation. We find the general solution to the homogeneous equation first:

$y''+y=0$ has characteristic equation roots i and $-i$, so the general solution is $y_h(t) = C_1 e^{it} + C_2 e^{-it}$.
We ~~will~~ can leave it in this complex form since ~~it~~ is maybe easier to differentiate. ~~a~~ Since $2i$ is not a root of the characteristic equation, we can write the solution in the form $y_p(t) = \cancel{A_1 e^{2it} + A}.\ A_1 \cos 2t + A_2 \sin 2t$.

$y_p' = -2A_1 \sin 2t + 2A_2 \cos 2t$

$y_p'' = -4A_1 \cos 2t - 4A_2 \sin 2t$.

Therefore,

$$y_p'' + y_p = -3A_1 \cos(2t) - 3A_2 \sin 2t = \sin 2t,$$

whence $-3A_1 = 0$, $-3A_2 = 1$, whence $A_1 = 0$, $A_2 = -1/3$.

Thus, the general solution is:

$$y_g(t) = C_1 e^{it} + C_2 e^{-it} - \frac{1}{3}\sin(2t)$$

$$y(0) = \boxed{C_1 + C_2 = 0}.$$

$$y'(t) = iC_1 e^{it} - iC_2 e^{-it} - \frac{2}{3}\cos(2t)$$

$y'(0) = iC_1 - iC_2 - \frac{2}{3} = 0 \Rightarrow \boxed{C_1 - C_2 = \frac{-2}{3}i}$, adding these equations gives:

$$2C_1 = -\tfrac{2}{3}i \Rightarrow C_1 = -\frac{1}{3}i \Rightarrow C_2 = \frac{1}{3}i \Rightarrow$$

$$y(t) = \frac{-1}{3}ie^{it} + \frac{1}{3}ie^{-it} - \frac{1}{3}\sin(2t)$$

$$= \frac{2}{3}\left[\frac{e^{it} - e^{-it}}{2i}\right] - \frac{1}{3}\sin(2t) = \boxed{\frac{2}{3}\sin(t) - \frac{1}{3}\sin(2t)}$$