① $\mathcal{L}(te^t) = \displaystyle\int_{t=0}^{\infty} e^{-st}\cdot te^t\,dt = \int_{t=0}^{\infty} \overbrace{e^{-t(s-1)}}^{dv}\,\overbrace{t}^{u}\,dt$

$$= \frac{-1}{s-1}e^{-t(s-1)}t\Big|_{t=0}^{\infty} + \frac{1}{s-1}\int_{t=0}^{\infty} e^{-t(s-1)}\,dt$$

$$= \frac{-1}{(s-1)^2}e^{-t(s-1)}\Big|_{t=0}^{\infty} = \frac{1}{(s-1)^2}.$$

Alternately, one can use formulas 14 and 3 of the table.

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② $\mathcal{L}(y'') - \mathcal{L}(y) = \mathcal{L}(e^{2t})$

$s^2 Y - sy(0) - y'(0) - Y = \frac{1}{s-2}.$

$(s^2-1)Y = \frac{1}{s-2} \Rightarrow Y(s) = \frac{1}{(s^2-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s-2}.$

we get $1 = A(s+1)(s-2) + B(s-1)(s-2) + C(s-1)(s+1).$

Plugging in $s=1$ gives $1 = A(2)(-1) \Rightarrow A = \frac{-1}{2}.$

Plugging in $s=-1$ gives $1 = B(-2)(-3) \Rightarrow B = \frac{1}{6}.$

Plugging in $s=2$ gives $1 = C(1)(3) \Rightarrow C = 1/3.$

Therefore,

$$Y(s) = \frac{-1}{2}\left(\frac{1}{s-1}\right) + \frac{1}{6}\left(\frac{1}{s+1}\right) + \frac{1}{3}\left(\frac{1}{s-2}\right).$$

Therefore,

$$y(t) = \frac{-1}{2}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + \frac{1}{6}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right)$$

$$= \frac{-1}{2}e^{t} + \frac{1}{6}e^{-t} + \frac{1}{3}e^{2t}$$

from formula ② in the table.

(3) As in problem (2), we have:

$$s^2 Y - Y = \mathcal{L}(u_1(t)) \Leftrightarrow (s^2-1)Y = \frac{e^{-s}}{s}$$

$$\Rightarrow Y(s) = e^{-s}\left[\frac{1}{s(s^2-1)}\right] = e^{-s}\left[\frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}\right]$$

Computing A, B, and C:

$$1 = A(s-1)(s+1) + B(s)(s+1) + C(s)(s-1)$$

Plugging in $s=0$:

$$1 = A(-1)(1) \Rightarrow A = -1.$$

Plugging in $s=+1$: $1 = B(1)(2) \Rightarrow B = \frac{1}{2}$

Plugging in $s=-1$: $1 = C(-1)(-2) \Rightarrow C = \frac{1}{2}$.

We thus have

$$Y(s) = -1\left(e^{-s}\cdot\frac{1}{s}\right) + \frac{1}{2}\left(e^{-s}\cdot\frac{1}{s-1}\right) + \frac{1}{2}\left(e^{-s}\cdot\frac{1}{s+1}\right)$$

whence

$$y(t) = -1u_1(t) + \frac{1}{2}u_1(t)e^{t-1} + \frac{1}{2}u_1(t)e^{-(t-1)}$$

$$= u_1(t)\left\{-1 + \overset{\cosh}{\cancel{\sinh}}(t-1)\right\}$$