Math. 350-03, 2007/11/21 - Fourth exam answers ①

① $y = \sum_{n=0}^{\infty} a_n (x-1)^n$. $y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$.

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$.

$m = n-1$

$x y'' = (x-1) y'' + y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-1} + \sum_{m=1}^{\infty} (m+1)(m) a_{m+1} (x-1)^{m-1}$

Thus, $x y'' + y' = a_1 + 2a_2 + \sum_{n=2}^{\infty} n^2 a_n (x-1)^{n-1} + (n+1) n a_{n+1} (x-1)^{n-1}$

$= a_1 + 2a_2 + \sum_{n=2}^{\infty} \left[n^2 a_n + (n+1)(n) a_{n+1} \right] (x-1)^{n-1}$.

$y(1) = a_0 = 0$

$y'(1) = a_1 = 1$.

$a_1 + 2a_2 = 0 \Rightarrow a_2 = \frac{-a_1}{2} = \frac{-1}{2}$.

$n \geq 2$: $n^2 a_n + (n+1)(n) a_{n+1} = 0 \Rightarrow a_{n+1} = \frac{-a_n n}{n+1}$.

Thus: $a_3 = \frac{-2a_2}{3} = \frac{1}{3}$; $a_4 = \frac{-3a_3}{4} = \frac{-1}{4}$; $a_5 = \frac{-4a_4}{5} = \frac{1}{5}$.

Thus, the approximation to y of degree 5 is:

$$\boxed{(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5}}$$

2 $y(x) = \sum_{n=0}^{\infty} a_n x^n$; $xy'(x) = x \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^n$.

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \underset{m=n-2}{=} \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m$$

Thus $y'' + xy' + y = 2a_2 + a_0 + \sum_{n=1}^{\infty} [a_n + n a_n + (n+2)(n+1) a_{n+2}] x^n$

On the other hand,

$$\sin(x) = x - x^3/6 + x^5/120 + \cdots$$

Thus, $a_0 = 1$, $a_1 = 0$, $2a_2 + a_0 = 0 \Rightarrow a_2 = -a_0/2 = -1/2$

The general left-hand side is:

$$(n+1) a_n + (n+2)(n+1) a_{n+2}.$$

for $n=1$: $2a_1 + 6a_3 = 1$, i.e. $a_3 = (1 - 2a_1)/6 = (1+0)/6 = \cancel{1/3}\ 1/6$.

for $n=2$: $3a_2 + 12a_4 = 0$, i.e. $a_4 = \frac{-3a_2}{12} = \cancel{3/2}\ +1/8$

for $n=3$: $4a_3 + 20a_5 = \cancel{0}\ -1/6$, i.e. $a_5 = (-1/6 - 4a_3)/20$

$$= (-1/6 + -4/6)/20$$

$$= \frac{-5}{6(20)} = \frac{-1}{24}.$$

Thus, the degree-5 approximation to the solution is:

$$\boxed{y(x) \approx 1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cancel{\tfrac{3}{12}}\,x^4 - \frac{1}{24}x^5}$$

$+1/8$