1(a) An integrating factor is $u(x) = e^{\int \frac{dx}{x}} = e^{\ln x} = x$.

We thus have: $(xy)' = x^2 \Rightarrow xy = \frac{x^3}{3} + C$

$\Rightarrow y = \frac{x^2}{3} + \frac{C}{x}$. $y(1) = 1 \Rightarrow 1 = \frac{1}{3} + C \Rightarrow C = \frac{2}{3}$.

$$\boxed{\therefore y(x) = \frac{x^2}{3} + \frac{2}{3}\left(\frac{1}{x}\right)}$$

1b The characteristic equation is $\alpha^2 + 4\alpha = 0 \Rightarrow \alpha = \pm 2i$, so the general solution to the homogeneous equation is: $y_h(t) = C_1 \cos(2t) + C_2 \sin(2t)$.

Since $\cos(2t)$ is a solution to the homogeneous equation, we need to assume a solution of the form $At\cos 2t + Bt\sin 2t$.

$$y_p(t) = At\cos 2t + Bt\sin 2t$$

$$y_p'(t) = A\cos 2t - 2At\sin 2t + B\sin 2t + 2Bt\cos 2t$$

$$y_p'' = -2A\sin 2t - 2A\sin 2t - 4At\cos 2t + 2B\cos 2t + 2B\cos 2t - 4Bt\sin 2t$$

$$= -4A\sin 2t + 4B\cos 2t - 4At\cos 2t - 4Bt\sin 2t.$$

Thus,

$y_p'' + 4y_p = -4A\sin 2t + 4B\cos 2t = \cos 2t \Rightarrow$

$A = 0$, $B = 1/4$, so the general solution is:

$$y_g(t) = C_1\cos 2t + C_2\sin 2t + \frac{1}{4}t\sin 2t$$

$$y_g(0) = C_1 \cancel{+C_2} = 0$$

$$y_g'(t) = -2C_1\sin 2t + 2C_2\cos 2t + \frac{1}{4}\cancel{t}\sin 2t + \frac{1}{2}t\cos 2t$$

$$y_g'(0) = 2C_2 = 0 \Rightarrow C_1 = C_2 = 0$$

$$\boxed{\therefore y(t) = \frac{1}{4}t\sin 2t}$$

(2) $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$. $(x-1)y = \sum_{n=0}^{\infty} a_n (x-1)^{n+1} = \sum_{n=1}^{\infty} a_{n-1} (x-1)^n$.

$y'(x) = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n$

$y' + (x-1)y = a_0 + \sum_{n=1}^{\infty} [(n+1)a_{n+1} + a_{n-1}](x-1)^n = (x-1)$

$a_0 = 0$, and

for $n=1$: $2a_2 + a_0 = 1 \Rightarrow a_2 = \frac{1}{2}$.

for $n>1$: $a_{n+1} = \frac{-a_{n-1}}{n+1}$.

$a_3 = 0$, $a_4 = -a_2/4 = -1/8$. $a_5 = 0$

$\therefore y(x) = \frac{1}{2}(x-1)^2 - \frac{1}{8}(x-1)^4$

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(3) $\mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}(e^{-t})$.

$sY - y(0) + Y = \frac{1}{s+1}$

$(s+1)Y = \frac{1}{s+1}$

$Y = \frac{1}{(s+1)^2}$.

Now, using (11) from the table, we get:

$\boxed{y(t) = te^{-t}}$.