1 (a) $y' = -\frac{1}{3}y + 1$. $y' + \frac{1}{3}y = +1$.

$u = e^{+\int \frac{1}{3}dt} = Ke^{t/3}$.

$(e^{t/3}y)' = e^{t/3}$

$e^{t/3}y = 3e^{t/3} + C$

$y = 3 + Ce^{-t/3}$

(b) An equilibrium occurs where $-\frac{1}{3}y + 1 = 0$, that is, where $\boxed{y = 3}$

(c) The equilibrium is stable, since $y' < 0$ for $y > 3$ and $y' > 0$ for $y < 3$.

(d) $y(0) = C + 3 = 1 \Rightarrow y(t) = 3 - 2e^{-t/3}$.

(e) Yes, since $e^{-t/3} \to 0$ as $t \to \infty$.

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2 $y'' + 4y' + 4 = 0$, $y(0) = 1$, $y'(0) = 0$.

The characteristic equation is $\alpha^2 + 4\alpha + 4 = (\alpha + 2)^2 = 0$.
Since $\alpha = -2$ is a double root, the general solution is $y(t) = c_1e^{-2t} + c_2te^{-2t}$.

$y(0) = c_1 = 1$.

$y'(t) = -2c_1e^{-2t} + c_2e^{-2t} - 2c_2te^{-2t}$, so

$y'(0) = -2c_1 + c_2 = -2 + c_2 = 0 \Rightarrow c_2 = 2$.

Thus, the solution to the initial value problem is:

$\boxed{y(t) = e^{-2t} + 2te^{-2t}}$

3 $y(x) = \sum_{n=0}^{\infty} a_n x^n$. $y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$.

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$, so $x^2 y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^n$.

Thus, $x^2 y'' + y' + y = \sum_{n=2}^{\infty} [n(n-1)a_n + (n+1)a_{n+1} + a_n] x^n + a_0 + a_1 x + a_1 + 2a_2 x$

$= \sum_{n=2}^{\infty} [(n^2-n+1) a_n + (n+1) a_{n+1}] x^n + (a_0 + a_1) + (a_1 + 2a_2) x = 1$.

Equating coefficients gives:

$\underline{n=0}$: $a_0 = 0, a_1 = 1$ (from the initial values)

$\underline{n=1}$: $a_1 + 2a_2 = 0 \Rightarrow a_2 = -1/2$.

$n \geq 2$: $a_{n+1} = \dfrac{-(n^2-n+1)}{n+1} a_n$

$\underline{n=2}$: $a_3 = -a_2 = \frac{1}{2}$

$\underline{n=3}$: $a_4 = -\frac{7}{4} a_3 = -\frac{7}{8}$

$\underline{n=4}$: $a_5 = -\frac{13}{5} a_4 = \frac{91}{40}$

$\boxed{\therefore\ y(x) \approx x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{7}{8}x^4 + \frac{91}{40}x^5}$

4 $y' = -\alpha y$, where y is the amount of uranium remaining.

Thus $y(t) = y_0 e^{-\alpha t}$. $y(4.5\times10^9) = \frac{1}{2}y_0 = y_0 e^{-4.5\times10^9\alpha}$

$$\Rightarrow \log_e\left(\tfrac{1}{2}\right) = -\ln(2) = -4.5\times10^9\alpha \Rightarrow \alpha = \frac{\ln(2)}{4.5\times10^9} \approx 1.5403\times10^{-10}$$

$$\Rightarrow y(t) = y_0\cdot 2^{-\left(\frac{t}{4.5\times10^9}\right)}$$

If T is the age of the stratum, we get

$$\frac{y(T)}{y_0} \approx 0.9772 \approx 2^{-\left(\frac{T}{4.5\times10^9}\right)}.$$

This gives $\ln(0.9772) = \frac{-T}{4.5\times10^9}\ln(2)$

$$\text{or } T = -\frac{4.5\times10^9\ln(0.9772)}{\ln(2)} = -\frac{4.5\times10^9\log_{10}(0.9772)}{\log_{10}(2)}$$

$\approx$ 1.4973×10^8 years, that is, about ≈ 150 million years.

$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(f)$, where $f(t) = u_\pi(t)\sin(2(t-\pi))$

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = e^{-\pi s}\left(\frac{2}{s^2+4}\right)$$

$$Y(s)(s^2+1) - s = e^{-\pi s}\left(\frac{2}{s^2+4}\right)$$

$$\Rightarrow Y(s) = \frac{s}{s^2+1} + e^{-\pi s}\left[\frac{2}{(s^2+1)(s^2+4)}\right].$$

Using partial fractions, we get:

$$\frac{2}{(s^2+1)(s^2+4)} = \frac{A}{s^2+1} + \frac{B}{s^2+4} \Rightarrow 2 = A(s^2+4) + B(s^2+1)$$
$$= (A+B)s^2 + (4A+B)$$

$A + B = 0 \Rightarrow B = -A \qquad 4A + B = 3A = 2 \Rightarrow A = 2/3, \; B = \frac{-2}{3}$

$$\Rightarrow \frac{2}{(s^2+1)(s^2+4)} = \frac{2/3}{s^2+1} - \frac{2/3}{s^2+4} = \frac{2}{3}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}\left(\frac{2}{s^2+4}\right)$$

Thus, $y(t) = \mathcal{L}^{-1}(s) = \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) + \frac{2}{3}\mathcal{L}^{-1}\left(e^{-\pi s}\left(\frac{1}{s^2+1}\right)\right)$

$$-\frac{1}{3}\mathcal{L}^{-1}\left(e^{-\pi s}\cdot\frac{2}{s^2+4}\right)$$

$$= \boxed{\cos(t) + \frac{2}{3}u_\pi(t)\sin(t-\pi) - \frac{1}{3}u_\pi(t)\sin(2(t-\pi))}$$