

1 a  $y' = -\frac{1}{3}y + 1$ .  ~~$y' + \frac{1}{3}y = 1$~~

$$u = e^{\int \frac{1}{3} dt} = Ke^{t/3}$$

$$(e^{t/3}y)' = e^{t/3}$$

$$e^{t/3}y = \frac{3}{1}e^{t/3} + C$$

$$y = 3 + Ce^{-t/3}$$

b An equilibrium occurs where  $-\frac{1}{3}y + 1 = 0$ ,  
that is, where  $y = 3$

c The equilibrium is stable, since  $y' < 0$  for  $y > 3$   
and  $y' > 0$  for  $y < 3$ .

d  $y(0) = C + 3 = 1 \Rightarrow y(t) = 3 - 2e^{-t/3}$

e Yes, since  $e^{-t/3} \rightarrow 0$  as  $t \rightarrow \infty$ .

2  $y'' + 4y' + 4 = 0$ ,  $y(0) = 1$ ,  $y'(0) = 0$ .

The characteristic equation is  $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0$ .

Since  $\lambda = -2$  is a double root, the general solution  
is  $y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$ .

$$y(0) = c_1 = 1.$$

$$y'(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}, \text{ so}$$

$$y'(0) = -2c_1 + c_2 = -2 + c_2 = 0 \Rightarrow c_2 = 2.$$

Thus, the solution to the initial value problem is:

$$y(t) = e^{-2t} + 2t e^{-2t}$$

$$\textcircled{3} \quad y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n,$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}, \quad \text{so } x^2 y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^n.$$

$$\begin{aligned} \text{Thus, } x^2 y'' + y' + y &= \sum_{n=2}^{\infty} [n(n-1) a_n + (n+1) a_{n+1} + a_n] x^n + a_0 + a_1 x + a_1 + 2a_2 x \\ &= \sum_{n=2}^{\infty} [(n^2 - n + 1) a_n + (n+1) a_{n+1}] x^n + (a_0 + a_1) + (a_1 + 2a_2) x = 1. \end{aligned}$$

Equating coefficients gives:

$$n=0: \quad a_0 = 0, a_1 = 1 \quad (\text{from the initial values})$$

$$n=1: \quad a_1 + 2a_2 = 0 \Rightarrow a_2 = -\frac{1}{2}.$$

$$n \geq 2: \quad a_{n+1} = -\frac{(n^2 - n + 1) a_n}{n+1}$$

$$n=2: \quad a_3 = -a_2 = \frac{1}{2}$$

$$n=3: \quad a_4 = -\frac{7}{4} a_3 = -\frac{7}{8}$$

$$n=4: \quad a_5 = -\frac{13}{5} a_4 = \frac{91}{40}$$

$$\therefore y(x) \approx x - \frac{1}{2} x^2 + \frac{1}{2} x^3 - \frac{7}{8} x^4 + \frac{91}{40} x^5$$

40	31	902453	WALSH	WALTER
39	30	90222	WALSH	WALTER
38	29	90275	WALSH	WALTER
37	28	90200	WALSH	WALTER
36	27	90201	WALSH	WALTER
35	26	90202	WALSH	WALTER
34	25	90203	WALSH	WALTER
33	24	90204	WALSH	WALTER
32	23	90205	WALSH	WALTER
31	22	90206	WALSH	WALTER
30	21	90207	WALSH	WALTER
29	20	90208	WALSH	WALTER
28	19	90209	WALSH	WALTER
27	18	90210	WALSH	WALTER
26	17	90211	WALSH	WALTER
25	16	90212	WALSH	WALTER
24	15	90213	WALSH	WALTER
23	14	90214	WALSH	WALTER
22	13	90215	WALSH	WALTER
21	12	90216	WALSH	WALTER
20	11	90217	WALSH	WALTER
19	10	90218	WALSH	WALTER
18	9	90219	WALSH	WALTER
17	8	90220	WALSH	WALTER
16	7	90221	WALSH	WALTER
15	6	90222	WALSH	WALTER
14	5	90223	WALSH	WALTER
13	4	90224	WALSH	WALTER
12	3	90225	WALSH	WALTER
11	2	90226	WALSH	WALTER
10	1	90227	WALSH	WALTER
9	0	90228	WALSH	WALTER
8	0	90229	WALSH	WALTER
7	0	90230	WALSH	WALTER
6	0	90231	WALSH	WALTER
5	0	90232	WALSH	WALTER
4	0	90233	WALSH	WALTER
3	0	90234	WALSH	WALTER
2	0	90235	WALSH	WALTER
1	0	90236	WALSH	WALTER
0	0	90237	WALSH	WALTER

(4)  $y' = -\alpha y$ , where  $y$  is the amount of uranium remaining.

Thus  $y(t) = y_0 e^{-\alpha t}$ .  $y(4.5 \times 10^9) = \frac{1}{2} y_0 = y_0 e^{-4.5 \times 10^9 \alpha}$

$\Rightarrow \log_{\frac{1}{2}}\left(\frac{1}{2}\right) = -\ln(2) = -4.5 \times 10^9 \alpha \Rightarrow \alpha = \frac{\ln(2)}{4.5 \times 10^9} \approx 1.5403 \times 10^{-10}$

$\Rightarrow y(t) = y_0 \cdot 2^{-\left(\frac{t}{4.5 \times 10^9}\right)}$

If  $T$  is the age of the stratum, we get

$\frac{y(T)}{y_0} \approx 0.9772 \approx 2^{-\left(\frac{T}{4.5 \times 10^9}\right)}$

This gives  $\ln(0.9772) = \frac{-T}{4.5 \times 10^9} \ln(2)$

or  $T = \frac{-4.5 \times 10^9 \ln(0.9772)}{\ln(2)} = \frac{-4.5 \times 10^9 \log_{10}(0.9772)}{\log_{10}(2)}$

$\approx 1.4973 \times 10^8$  years, that is, about  
 $\approx 150$  million years.

$$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(f), \text{ where } f(t) = u_{\pi}(t) \sin(2(t-\pi))$$

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = e^{-\pi s} \left( \frac{2}{s^2+4} \right)$$

$$\mathcal{L} Y(s) (s^2+1) - s = e^{-\pi s} \left( \frac{2}{s^2+4} \right)$$

$$\Rightarrow Y(s) = \frac{s}{s^2+1} + e^{-\pi s} \left[ \frac{2}{(s^2+1)(s^2+4)} \right].$$

Using partial fractions, we get:

$$\frac{2}{(s^2+1)(s^2+4)} = \frac{A}{s^2+1} + \frac{B}{s^2+4} \Rightarrow 2 = A(s^2+4) + B(s^2+1)$$

$$= (A+B)s^2 + (4A+B)$$

$$A+B=0 \Rightarrow B=-A \quad 4A+B=3A=2 \Rightarrow A=\frac{2}{3}, B=-\frac{2}{3}$$

$$\Rightarrow \frac{2}{(s^2+1)(s^2+4)} = \frac{2/3}{s^2+1} - \frac{2/3}{s^2+4} = \frac{2}{3} \left( \frac{1}{s^2+1} \right) - \frac{1}{3} \left( \frac{2}{s^2+4} \right)$$

$$\text{Thus, } y(t) = \mathcal{L}^{-1}(s) = \mathcal{L}^{-1} \left( \frac{s}{s^2+1} \right) + \frac{2}{3} \mathcal{L}^{-1} \left( e^{-\pi s} \left( \frac{1}{s^2+1} \right) \right) - \frac{1}{3} \mathcal{L}^{-1} \left( e^{-\pi s} \left( \frac{2}{s^2+4} \right) \right)$$

$$= \left[ \cos(t) + \frac{2}{3} u_{\pi}(t) \sin(t-\pi) - \frac{1}{3} u_{\pi}(t) \sin(2(t-\pi)) \right]$$