(a) $y(x) = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, so $2xy' = \sum_{n=1}^{\infty} (2na_n)x^n$.

$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$.

Thus, $y'' + 2xy' + y = \sum_{n=\cancel{0}}^{\infty} [(n+2)(n+1)a_{n+2} + 2na_n + a_n]x^n \cancel{=0}$

$+ a_0 + 2a_2 = 0.$

Thus, $a_0 + 2a_2 = 0 \Rightarrow a_2 = -\frac{1}{2}a_0$, and, for $n \geq 1$,

$[(n+2)(n+1)a_{n+2} + (2n+1)a_n = 0 \Rightarrow$

$$a_{n+2} = -\frac{(2n+1)a_n}{(n+2)(n+1)} \text{ for } n \geq 1.$$

Thus, $a_0 = 1$, $a_1 = 2$, $a_2 = \frac{-1}{2}$,

$$a_3 = -\frac{(3)a_1}{(3)(2)} = -\frac{1}{2}a_1 = -1.$$

$$a_4 = -\frac{5a_2}{(4)(3)} = -\frac{5\left(\frac{-1}{2}\right)}{4(3)} = \frac{5}{24}.$$

The degree 4 Taylor approximation to the solution is thus:

$$\boxed{p_4(x) = 1 + 2x - \frac{1}{2}x^2 - x^3 + \frac{5}{24}x^4}$$

(b) $y(x) = \sum_{n=0}^{\infty} a_n x^n$; $y' \sim \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$.

$(x-1)y \sim \sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1}x^n - \sum_{n=0}^{\infty} a_n x^n$,

so $y' + (x-1)y = a_1 - a_0 + \sum_{n=1}^{\infty}[a_{n-1} - a_n + (n+1)a_{n+1}]x^n = 0$

$\boxed{a_0 = 1}$, $a_1 - a_0 = 0 \Rightarrow \boxed{a_1 = 1}$.

For $n \geq 1$, $a_{n+1} = \frac{a_n - a_{n-1}}{n+1}$. Thus,

$a_2 = \frac{1-1}{2} = \boxed{0 = a_2}$, $a_3 = \frac{a_2 - a_1}{3} = \frac{0-1}{3} = \boxed{\frac{-1}{3} = a_3}$.

$$a_4 = \frac{a_3 - a_2}{4} = \frac{\frac{-1}{3} - 0}{4} = \frac{-1}{12}.$$

Thus, $\boxed{y(x) \approx 1 + x - \frac{1}{3}x^3 - \frac{1}{12}x^4}$

(c) $y'' \sim \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$; $xy \sim \sum_{n=1}^{\infty} a_{n-1}x^n$

$$y''+xy \sim 2a_2 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_{n-1}\right]x^n$$
$$= x - \frac{x^3}{3!} + \cdots$$

$\boxed{a_0 = 1}$ $\boxed{a_1 = 0}$ $2a_2 = 0 \Rightarrow \boxed{a_2 = 0}$

$6a_3 + a_0 = 1 \Rightarrow a_3 = \frac{1-1}{6} = \boxed{0 = a_3}$.

$(4)(3)a_4 + a_1 = 0 \Rightarrow a_4 = \frac{-0}{12} = \boxed{0 = a_4}$.

Thus, $\boxed{y(x) \approx P_4(x) = \text{[redacted]}}$