(1) The characteristic equation is

$$\alpha^2+4\alpha+5=0,$$ whose roots are

$$\alpha = \frac{-4}{2} \pm \frac{\sqrt{16-4(1)(5)}}{2} = -2 \pm \frac{\sqrt{-4}}{2} = -2 \pm i.$$

The general solution can therefore be written as

$$\boxed{y(t) = e^{-2t}(C_1 \cos t + C_2 \sin t)}$$

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(2)
$$e^{(-2+i)t} = e^{-2t}(\cos t + i \sin t)$$
$$-\left[e^{(-2-i)t} = e^{-2t}(\cos t - i \sin t)\right]$$

$$e^{(-2+i)t} - e^{(-2-i)t} = e^{-2t}(2i \sin t)$$

so
$$\boxed{\frac{e^{(-2+i)t} - e^{(-2-i)t}}{2i} = e^{-2t}\sin t}$$

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(3) The characteristic equation is $\alpha^2+4=0$, with roots $\pm 2i$, so the general solution to the homogeneous equation is:

$$y_h(t) = C_1\cos 2t + C_2 \sin 2t.$$

Since the forcing term is also a solution to the corresponding homogeneous equation, we assume a solution of the form

$$y_p(t) = At\cos 2t + Bt \sin 2t.$$

Plugging in:

$$y_p'(t) = A\cos 2t - 2At\sin 2t + B\sin 2t + 2Bt\cos 2t$$

$$y_p''(t) = -2A\sin 2t - 2A\sin 2t - 4At\cos 2t + 2B\cos 2t + 2B\cos 2t - 4Bt\sin 2t$$

$$= -4A\sin 2t + 4B\cos 2t - 4At\cos 2t - 4Bt\sin 2t$$

so $y_p''+4y_p = -4A\sin 2t + 4B\cos 2t = \sin 2t$.

This gives $-4A=1 \Rightarrow A=-\frac{1}{4}$, and $4B=0 \Rightarrow B=0$,

so $\boxed{y_p(t) = -\frac{1}{4}t\cos 2t}$

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The general solution is thus:

$$y(t) = C_1 \cos 2t + C_2 \sin 2t - \frac{1}{4} t \cos 2t.$$

Plugging into the initial conditions:

$$y(0) = C_1 = 0$$

$$y'(t) = -2C_1 \sin 2t + 2C_2 \cos 2t - \frac{1}{4}\cos(2t) + \frac{1}{2} t \sin 2t$$

$$y'(0) = 2C_2 - \frac{1}{4} = 0 \Rightarrow C_2 = \frac{1}{8}.$$

This gives: $\boxed{y(t) = \frac{1}{8}\sin 2t - \frac{1}{4} t \cos 2t}$