Math. 350-02, Spring, 2009, exam 3b (makeup)

① The solution to the homogeneous equation is:

$$y_h(t) = C_1 e^{-2t} + C_2 t e^{-2t}.$$

Since $e^{-2t}$ and $te^{-2t}$ are solutions to the homogeneous equation, we assume a particular solution of the form

$$y_p(t) = At^2(te^{-2t}) = At^3 e^{-2t}.$$

Plugging in, we obtain:

$$y_p' = A(3t^2 e^{-2t} - 2t^3 e^{-2t})$$

$$y_p'' = A(6te^{-2t} - 12t^2 e^{-2t} + 4t^3 e^{-2t})$$

so $y_p'' + 4y_p' + 4y_p = A(6te^{-2t}) = te^{-2t}.$

This gives $6A = 1 \Rightarrow A = \frac{1}{6} \Rightarrow y_p(t) = \frac{1}{6}t^3 e^{-2t}.$

Thus, the general solution is:

$$y(t) = C_1 e^{-2t} + C_2 te^{-2t} + \frac{1}{6}t^3 e^{-2t}.$$

Plugging the initial conditions in gives:

$$y(0) = C_1 + 0 + 0 = 0 \Rightarrow C_1 = 0$$

$$y'(t) = -2C_1 e^{-2t} \cancel{te} - 2C_2 te^{-2t} + C_2 e^{-2t} + \frac{1}{2}t^2 e^{-2t} - \frac{1}{3}t^2 e^{-2t}.$$

$$y'(0) = 0 - 0 + C_2 = 1 \Rightarrow C_2 = 1.$$

Thus,

$$\boxed{y(t) = te^{-2t} + \frac{1}{6}t^3 e^{-2t}}.$$

② $\boxed{\cos(3t) + i\sin(3t) = e^{3it}}$