Math. 350-02, Monday, March 30, 2009, Fourth Exam Answers, page 1

(1)

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\cos(x)$	0
1	$-\sin(x)$	-1
2	$-\cos(x)$	0
3	$\sin(x)$	1
4	$\cos(x)$	0
5	$-\sin(x)$	-1

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(x-x_0)^n.$$

$$\cos(x) \approx 0 - \frac{1}{1!}(x-\pi/2) + \frac{0}{2!}(x-\pi/2)^2 + \frac{1}{3!}(x-\pi/2)^3 + \frac{0}{4!}(x-\pi/2)^4 + \frac{(-1)}{5!}(x-\pi/2)^5$$

$$\boxed{\therefore \cos(x) \approx -(x-\pi/2) + \frac{1}{6}(x-\pi/2)^3 - \frac{1}{120}(x-\pi/2)^5}$$

---

(2) We assume $y(x) = \sum_{n=0}^{\infty} a_n (x)^n$, so $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

and $y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.

Thus, $y'' + x^2 y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x^2 \sum_{n=0}^{\infty} a_n x^n$

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2}$$

Substitute $m = n-2$ in the first sum and $m = n+2$ in the second sum to obtain:

$$y'' + x^2 y = \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m + \sum_{m=2}^{\infty} a_{m-2} x^m$$

$$= 2(1) a_2 + 3(2) a_3 x + \sum_{m=2}^{\infty} \left[(m+2)(m+1) a_{m+2} + a_{m-2}\right] x^m = 0$$

We thus obtain:

$2a_2 = 0$, $6a_3 = 0$,

$(m+2)(m+1) a_{m+2} + a_{m-2} = 0$, $m \geq 2$. $\Rightarrow a_{m+2} = \dfrac{-a_{m-2}}{(m+2)(m+1)}$, $m \geq 2$.

Thus, $a_0 = 1$, $a_1 = 2$, $a_2 = 0$, $a_3 = 0$, $a_4 = \dfrac{-a_0}{(4)(3)} = \dfrac{-1}{12}$, $a_5 = \dfrac{-a_1}{(5)(4)} = \dfrac{-1}{10}$.

The polynomial is thus:

$$\boxed{y \approx P_5(x) = 1 + 2x - \frac{1}{12}x^4 - \frac{1}{10}x^5}$$