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1. The characteristic equation is: $r^2+4r+5=0$

$$r = \frac{-4}{2} \pm \frac{\sqrt{|16-20|}}{2}\, i = -2 \pm \frac{\sqrt{4}}{2}\, i = -2 \pm i,$$ so

the general solution can be written as:

$$y(t) = C_1 e^{-2t}\cos(t) + C_2 e^{-2t}\sin(t).$$

$$y(0) = \boxed{C_1 = 1}.$$

$$y'(t) = -2C_1 e^{-2t}\cos(t) - C_1 e^{-2t}\sin(t)$$
$$- 2C_2 e^{-2t}\sin(t) + C_2 e^{-2t}\cos(t),$$

so $y'(0) = -2C_1 + C_2 = -2 + C_2 = 0 \Rightarrow \boxed{C_2 = 2}$.

$$\boxed{\therefore y(t) = e^{-2t}\cos(t) + 2e^{-2t}\sin(t)}.$$

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2. $\cos(A-B) - \cos(A+B) = 2\sin(A)\sin(B)$

$$A - B = \frac{1000}{2\pi}t$$
$$A + B = \frac{1002}{2\pi}t$$
$$2A = \frac{2002}{2\pi}t$$

$$\Rightarrow A = \frac{1001}{2\pi}t,\quad B = \frac{1}{2\pi}t \Rightarrow$$

$$\therefore \cos\left(\frac{1000}{2\pi}t\right) - \cos\left(\frac{1002}{2\pi}t\right) = \boxed{2\sin\left(\frac{1001}{2\pi}t\right)\sin\left(\frac{1}{2\pi}t\right)}$$

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3. $A\cos(10^6t-\delta) = A\cos(10^6t)\cos(\delta) + A\sin(10^6t)\sin\delta$,

a) so $A\cos\delta = \frac{1}{\sqrt{2}}$, $A\sin\delta = \frac{1}{\sqrt{2}}$, so $\tan(\delta)=1$, and δ is a first-quadrant angle. This gives $\boxed{\delta = \pi/4}$

We also have $A^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1$, so $A=1$.

Therefore, $\frac{1}{\sqrt{2}}\cos(10^6t) + \frac{1}{\sqrt{2}}\sin(10^6t) = \cos(10^6t - \pi/4)$.

b) The amplitude is $\boxed{1}$.

d) The phase shift is $\boxed{\pi/4}$, corresponding to $45°$.

c) The period is $\boxed{\frac{2\pi}{10^6}}$.

4. The characteristic equation is:

$r^2+4=0 \Rightarrow r=\pm 2i$, so the homogeneous equation has general solution:

$y_h(t) = c_1\cos(2t) + c_2\sin(2t)$

We observe the forcing term is a solution to the homogeneous equation, so we need to multiply the form we would have assumed in undetermined coefficients by t. An appropriate form for this particular equation is: $y_p = At\sin(2t) + Bt\cos 2t$

(continued on next page)

(problem 4 continued)

$$y_p' = A\sin(2t) + 2At\cos(2t) + B\cos(2t) - 2Bt\sin(2t)$$

$$y_p'' = 2A\cos(2t) + 2A\cos(2t) - 4At\sin(2t)$$
$$\quad -2B\sin(2t) - 2B\sin(2t) - 4Bt\cos 2t$$
$$= 4A\cos 2t - 4B\sin(2t) - 4At\sin(2t) - 4Bt\cos(2t)$$

Thus, $y_p'' + 4y_p = 4A\cos 2t - 4B\sin 2t = \sin(2t)$

$\Rightarrow 4A = 0,\ A = 0;\ -4B = 1,\ B = \frac{-1}{4}$.

Thus, $y_p = \frac{-1}{4}t\cos 2t$ is a particular solution.

The general solution is thus:

$$y = C_1\cos(2t) + C_2\sin(2t) - \frac{1}{4}t\cos(2t).$$

$y(0) = \boxed{C_1 = 0}$

$y'(t) = 2C_2\cos(2t) - \frac{1}{4}\cos(2t) + \frac{1}{2}t\sin(2t)$

so $y'(0) = 2C_2 - \frac{1}{4} = 0$, ~~so~~ $\boxed{C_2 = \frac{1}{8}}$.

$$\boxed{\therefore\ y(t) = \frac{1}{8}\sin(2t) - \frac{1}{4}t\cos(2t)}$$