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1 (a) $a_n = \frac{1}{3^n}$. The radius of convergence is

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{\frac{1}{3^n}}{\frac{1}{3^{n+1}}} = \frac{3^{n+1}}{3^n} = \boxed{3}.$$

(b) The series converges in the interval

$|x-1|<3$, that is, $-2<x<4$.

at $x=-2$, the series becomes $\sum_{n=0}^{\infty}\frac{1}{3^n}(-3)^n = \sum_{n=0}^{\infty}(-1)^n$, not convergent, and at $x=4$, we have

$\sum_{n=0}^{\infty}\frac{3^n}{3^n} = \sum_{n=0}^{\infty}1$, not convergent. Hence, the interval of convergence is the open interval $\boxed{(-2,4)}$.

2 The first four terms of the series for e^u are:

$1+u+\frac{u^2}{2}+\frac{u^3}{6}$. Hence, the first four terms for the series for e^{2x} are:

$$1+(2x)+\frac{(2x)^2}{2}+\frac{(2x)^3}{6} = \boxed{1+2x+2x^2+\frac{4}{3}x^3}$$

3 $y(x) \sim \sum_{n=0}^{\infty}a_n x^n$, $y'(x) = \sum_{n=1}^{\infty}n a_n x^{n-1}$, $y''(x) = \sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}$.

$x^2y(x) = \sum_{n=0}^{\infty}a_n x^{n+2}$, $xy' = \sum_{n=1}^{\infty}n a_n x^n$, $y'' = \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m$.

$x^2y+xy'+y''$

$$= \sum_{m=2}^{\infty}a_{m-2}x^n + \sum_{n=1}^{\infty}n a_n x^n + \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m$$

$$= a_1x+2a_2+6a_3x+\sum_{n=2}^{\infty}[a_{n-2}+na_n+(n+2)(n+1)a_{n+2}]x^n = 0$$

$\Rightarrow 2a_2=0,\ a_1+6a_3=0,$

$n\ge 2$: $a_{n+2} = \frac{-1}{(n+2)(n+1)}[a_{n-2}+na_n]$.

$$a_0 = 1,\ a_1 = 1,\ a_2 = 0,\ a_3 = \frac{-1}{(3)(2)}(a_0 + 2a_2)$$

$$= \frac{-1}{6} a_0 = \frac{-1}{6}.$$

The first four terms of the series solution are thus:

$$\boxed{y(x) \approx 1 + x + 0x^2 - \frac{1}{6}x^3}$$