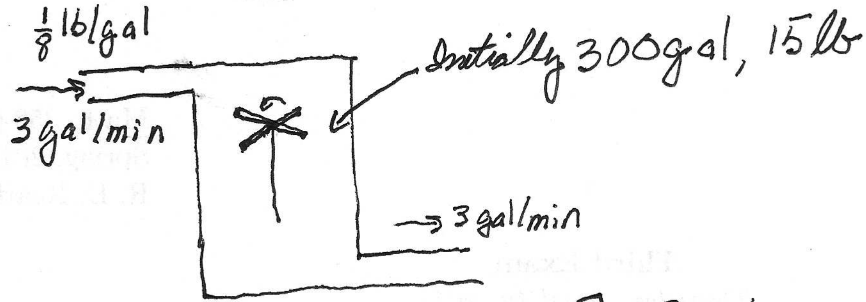
(1)



$[\text{Rate in}] = \left[\frac{1}{8}\ \text{lb/gal}\right]\left[3\ \text{gal/min}\right] = \frac{3}{8}\ \text{lb/min}$

$[\text{Rate out}] = \left[Q/300\ \text{lb/gal}\right]\left[3\ \text{gal/min}\right] = \frac{Q}{100}\ \text{lb/min}.$

so $\frac{dQ}{dt} = \frac{3}{8} - \frac{1}{100}Q.\quad Q' + .01Q = .375.\quad \mu = e^{\int .01\,dt} = e^{.01t}$

$(e^{.01t}Q)' = .375e^{.01t} \Rightarrow e^{.01t}Q = \frac{.375}{.01}e^{.01t} + C = 37.5e^{.01t} + C$

$\Rightarrow Q(t) = 37.5 + Ce^{-.01t}.\quad Q(0) = 15 = 37.5 + C \Rightarrow C = -22.5$

$\boxed{(a)\ Q(t) = 37.5 - 22.5e^{-.01t}}$.

(b) After a long period of time there will be approximately 37.5 pounds of salt in the tank.

2 The characteristic equation is:

$$r^2+8r+17=0 \Rightarrow r=-\frac{8}{2} \pm \frac{\sqrt{8^2-4(17)}}{2} = -4 \pm \frac{\sqrt{-4}}{2} = -4 \pm i,$$

so the general solution to the homogeneous equation is:

$$\boxed{y_h(t) = e^{-4t}\left[C_1\cos(t)+C_2\sin(t)\right].}$$

A particular solution is of the form:

$$y_p(t) = A\cos(t) + B\sin t$$

$$\begin{aligned} 17\,y_p &= 17A\cos(t) + 17B\sin(t) \\ 8\,y_p' &= 8B\cos(t) - 8A\sin(t) \\ y_p'' &= -A\cos(t) - B\sin(t) \end{aligned}$$

$$(16A+8B)\cos(t) + (-8A+16B)\sin(t) = \sin(t)$$

$$\Rightarrow \begin{cases} 16A+8B=0 \\ -8A+16B=1 \end{cases} \Rightarrow A=-\tfrac{1}{2}B \Rightarrow -8\left(-\tfrac{1}{2}B\right)+16B = 4B+16B = 20B = 1$$

$$\Rightarrow B=\frac{1}{20},\ A=\frac{-1}{40} \Rightarrow \boxed{y_p = \frac{-1}{40}\cos(t)+\frac{1}{20}\sin(t)}$$

Thus, the general solution is

$$y(t) = \frac{-1}{40}\cos(t)+\frac{1}{20}\sin(t)+e^{-4t}\left[C_1\cos(t)+C_2\sin(t)\right]$$

$$y(0)=1=\frac{-1}{40}+C_1 \Rightarrow \boxed{C_1=\frac{41}{40}}.$$

$$y'(t) = \frac{1}{40}\sin(t)+\frac{1}{20}\cos(t) - 4e^{-4t}\left[C_1\cos(t)+C_2\sin t\right] + e^{-4t}\left[-C_1\sin(t)+C_2\cos(t)\right]$$

$$y'(0) = \frac{1}{20}-4C_1+C_2 = \frac{1}{20}-\frac{41}{10}+C_2 = 0 \Rightarrow C_2 = \frac{82}{20}-\frac{2}{20} = \frac{80}{20} = \boxed{4=C_2}.$$

Thus,

(a) $$\boxed{y(t) = \frac{-1}{40}\cos(t)+\frac{1}{20}\sin(t)+e^{-4t}\left[\frac{41}{40}\cos(t)+4\sin(t)\right]}$$

(b) The steady-state solution is

$$\boxed{y(t) = \frac{-1}{40}\cos(t)+\frac{1}{20}\sin(t)}$$

3 $y=\sum_{n=0}^{\infty} a_n x^n$. $x^2 y=\sum_{n=0}^{\infty} a_n x^{n+2}=\sum_{n=2}^{\infty} a_{n-2}x^n$.

$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$, so $y'+x^2y=\cancel{a_0+a_1x}+\sum_{n=1}^{\infty} n a_n x^n+\sum_{n=2}^{\infty} a_{n-2}x^n$

$=a_1 x+\sum_{n=2}^{\infty}[n a_n+a_{n-2}]x^n=0$

$\Rightarrow \boxed{a_1=0}$, $n\geq 2$: $n a_n+a_{n-2}=0 \Rightarrow \boxed{a_n=\frac{-1}{n}a_{n-2}}$.

$\boxed{a_0=-1}$ $\boxed{a_1=0}$ $n=2$: $a_2=-\frac{1}{2}(-1)=\boxed{\frac{1}{2}=a_2}$.

$n=3$: $a_3=-\frac{1}{3}a_1=\boxed{0=a_3}$. $n=4$: $a_4=-\frac{1}{4}a_2=\frac{-1}{4}\left(\frac{1}{2}\right)=\boxed{\frac{-1}{8}=a_4}$.

$n=5$: $a_5=-\frac{1}{5}a_3=\boxed{0=a_5}$.

Thus, $y(x)\approx\boxed{P_5(x)=-1+\frac{1}{2}x^2-\frac{1}{8}x^4}$.

4 The DE is: $y''+4y'+3y=-u_2(t)+u_3(t)$,

so the Laplace transform is:

$$s^2Y-sy(0)-y'(0)+4sY-y(0)+3Y=-\frac{1}{s}e^{-2s}+\frac{1}{s}e^{-3s},$$

whence $(s^2+4s+3)Y-1=\frac{1}{s}\left[-e^{-2s}+e^{-3s}\right]$,

so $$Y=\frac{1}{s^2+4s+3}+\frac{1}{s(s^2+4s+3)}\left[-e^{-2s}+e^{-3s}\right].$$

We need to do 2 partial fraction decompositions:

$$\frac{1}{s^2+4s+3}=\frac{1}{(s+3)(s+1)}=\frac{A}{s+3}+\frac{B}{s+1}\Rightarrow 1=A(s+1)+B(s+3)=(A+B)s+(A+3B)$$

$$\Rightarrow A+3B=1$$
$$-[A+B=0]$$
$$2B=1\Rightarrow \boxed{B=1/2,\ A=-1/2}.$$

$$\frac{1}{s(s+3)(s+1)}=\frac{C}{s}+\frac{D}{s+3}+\frac{E}{s+1}\Rightarrow C(s+1)(s+3)+Ds(s+1)+Es(s+3)=1$$

$$\Rightarrow [C+D+E]s^2+[4C+D+3E]s+3C=1$$

$$\Rightarrow 3C=1\Rightarrow \boxed{C=\tfrac{1}{3}}\Rightarrow -\left[\tfrac{1}{3}+D+E=0\right]$$
$$\tfrac{4}{3}+D+3E=0$$
$$1+2E=0\Rightarrow \boxed{E=-\tfrac{1}{2}}\quad \tfrac{1}{3}+D-\tfrac{1}{2}=0\Rightarrow \boxed{D=\tfrac{1}{6}}$$

Thus, $$Y(s)=\frac{1}{2}\left(\frac{1}{s+3}\right)-\frac{1}{2}\left(\frac{1}{s+1}\right)+\left[\frac{1}{3}\frac{1}{s}+\frac{1}{6}\left(\frac{1}{s+3}\right)-\frac{1}{2}\left(\frac{1}{s+1}\right)\right]\left[-e^{-2s}+e^{-3s}\right],$$

so $$y(t)=\frac{1}{2}e^{-3t}-\frac{1}{2}e^{-t}-\left[\frac{1}{3}+\frac{1}{6}e^{-3(t-2)}-\frac{1}{2}e^{-(t-2)}\right]u_2(t)$$
$$+\left[\frac{1}{3}+\frac{1}{6}e^{-3(t-3)}-\frac{1}{2}e^{-(t-3)}\right]u_3(t)$$