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$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

$$y'' + xy' + x^2 y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 1$$

$\underline{n=0}$: $\boxed{2a_2 = 1}$; $\underline{n=1}$: $\boxed{6a_3 + a_1 = 0}$

$\underline{n \geq 2}$: $(n+2)(n+1)a_{n+2} + n a_n + a_{n-2} = 0$

$$\Rightarrow \boxed{a_{n+2} = -(n a_n + a_{n-2})/(n+2)(n+1)}.$$

Thus, $a_0 = y(0) = \boxed{0}$, $a_1 = y'(0) = \boxed{1 = a_1}$

$\boxed{a_2 = 1/2}$ $\quad a_3 = -(a_1/6) = \boxed{-1/6 = a_3}$

$\underline{n=2}$: $a_4 = -(2a_2 + a_0)/12 = -(2(1/2)+0)/12 = \boxed{-1/12 = a_4}$.

$\underline{n=3}$: $a_5 = -(3a_3 + a_1)/20 = -(3(-1/6)+1)/20$

$$= -\left(\tfrac{1}{2}\right)/20 = \boxed{\frac{-1}{40} = a_5}.$$

Thus, $\boxed{y(x) \sim x + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{40}x^5}$.

2) $\mathcal{L}(y''+3y'+2y)=\mathcal{L}(f)$

We write $f(t)=u_\pi(t)-u_{2\pi}(t)$, and we get

$$s^2Y - s + 3sY - 3 + 2Y = e^{-\pi s}\left(\frac{1}{s}\right) - e^{-2\pi s}\left(\frac{1}{s}\right)$$

and we solve for $Y(s)$:

$$(s^2+3s+2)Y = s+3 + (e^{-\pi s}-e^{-2\pi s})\left(\frac{1}{s}\right)$$

$$Y(s) = \frac{s+3}{(s+1)(s+2)} + \frac{1}{(s+1)(s+2)s}(e^{-\pi s}-e^{-2\pi s})$$

$$\frac{s+3}{(s+1)(s+2)} = \frac{A}{s+1}+\frac{B}{s+2} \Rightarrow s+3 = A(s+2)+B(s+1) = (A+B)s + (2A+B)$$

$$\Rightarrow A+B=1,\ 2A+B=3 \Rightarrow \boxed{A=2,\ B=-1}.$$

$$\frac{1}{s(s+1)(s+2)} = \frac{C}{s}+\frac{D}{s+1}+\frac{E}{s+2} \Rightarrow C(s+1)(s+2)+D(s)(s+2)+E(s)(s+1)=1$$

i.e. $1=(C+D+E)s^2+(3C+2D+E)s+(2C)$

so $2C=1 \Rightarrow \boxed{C=1/2}$ $\quad C+D+E=0,\ 3C+2D+E=0,$

That is, $\left.\begin{array}{l} D+E=-1/2 \\ 2D+E=\cancel{3/2}\ -3/2 \end{array}\right\} \Rightarrow \cancel{\boxed{D=1/2,\ E=-1}}\ \boxed{D=-1,\ E=+1/2}$

Thus $\cancel{\frac{1}{s(s+1)(s+2)}}$, $Y(s)=\frac{2}{s+1}-\frac{1}{s+2}+(e^{-\pi s}-e^{-2\pi s})\left(\frac{\frac{1}{2}}{s}-\frac{1}{s+1}+\frac{\frac{1}{2}}{s+2}\right)$

Using (2) and (13) of the table, we obtain:

$$\boxed{\begin{aligned} y(t) = 2e^{-t}-e^{-2t} &+ u_\pi(t)\left(\tfrac{1}{2}-e^{-(t-\pi)}+\tfrac{1}{2}e^{-2(t-\pi)}\right) \\ &- u_{2\pi}(t)\left(\tfrac{1}{2}-e^{-(t-2\pi)}+\tfrac{1}{2}e^{-2(t-2\pi)}\right)\end{aligned}}$$

The characteristic equation is:

$$r^2+3r+2=0 \Leftrightarrow (r+1)(r+2)=0, \text{ ie } r=-1 \text{ or } r=-2.$$

The general solution is thus

$$y(t) = c_1 e^{-t} + c_2 e^{-2t}$$

$$y(0) = \boxed{c_1 + c_2 = 1}$$

$$y'(t) = -c_1 e^{-t} - 2c_2 e^{-2t} = 0$$

so $y'(0) = \boxed{-c_1 - 2c_2 = 0}$.

Solving for c_1 and c_2 gives

$$c_1 + 2c_2 = 0$$
$$-[c_1 + c_2 = 1]$$
$$\boxed{c_2 = -1,\ c_1 = 2}$$

whence $\boxed{y(t) = 2e^{-t} - e^{-2t}}$.

This is the first part of the solution to Problem 2, as it should be.