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(a) $|A-\lambda I| = \begin{vmatrix} 4-\lambda & -1 \\ -1 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 - 1 = \lambda^2 - 8\lambda + 15 = 0$

for $\lambda = 3$ and $\lambda = 5$. When $\lambda = 3$,

$A - \lambda I = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, whose null space

is of the form $\left\{ v_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$, so an eigenvector corresponding

to $\lambda = 3$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Similarly, for $\lambda = 5$,

$A - \lambda I = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$, whose null space

is of the form $\left\{ v_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$, so an eigenvector

corresponding to $\lambda = 5$ is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(b) Since the eigenvectors form a basis for $\mathbb{R}^2$,

A is diagonalizable, and we have

$A = VDV^{-1}$, where $V = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$

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2 (a) We will use an elementary rotation.

We obtain $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$

Where $\alpha^2 + \beta^2 = 1$

$$3\beta + 4\alpha = 0 \Rightarrow \beta = -\frac{4}{3}\alpha \Rightarrow \alpha^2 + \left(\frac{16}{9}\right)\alpha^2 = 1$$

$\Rightarrow \left(\frac{25}{9}\right)\alpha^2 = 1 \Rightarrow \alpha = \pm \cancel{\frac{4}{3}}\frac{3}{5}$ If $\alpha = \cancel{\frac{4}{3}}\frac{3}{5}$, then $\beta = \frac{-4}{5}$,

and the transformation is $Q^T = \begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix}$

We obtain $Q^T A = R$, so $A = QR$, where $Q = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$

and $R = \begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 22/5 \\ 0 & 4/5 \end{bmatrix}$.

Check: $\begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}\begin{bmatrix} 5 & 22/5 \\ 0 & 4/5 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 3 & 2 \\ 4 & 4 \end{bmatrix}$.

(b) $Ax = Q(Rx) = b \Leftrightarrow Rx = Q^T b = \begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 7/5 \end{bmatrix}$

Thus, $\begin{bmatrix} 5 & 22/5 \\ 0 & 4/5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 7/5 \end{bmatrix}$

$\Rightarrow x_2 = \frac{5}{4}\left(\frac{7}{5}\right) = \boxed{\frac{7}{4}}$, $x_1 = \left(\frac{1}{5} - \frac{22}{5}\left(\frac{7}{4}\right)\right)\frac{1}{5}$

$= \left(\cancel{\frac{-15/10}{20}}\right)\cdot\frac{1}{5} = \boxed{\cancel{-\frac{3}{2}}}$.

$\left(\frac{-15}{2}\right)\left(\frac{1}{5}\right) = -3/2$

$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cancel{7/4} \\ \cancel{-3/4} \end{bmatrix}}$ $\cancel{-3/2}$, $\begin{bmatrix} -3/2 \\ 7/4 \end{bmatrix}$.