# A New Interval Selection Technique for Global Optimization

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## Problem

Consider the bound constrained global optimization problem

$$\min_{x \in X} f(x)$$

where the $n$-dimensional interval $X$ is the search region, and $f(x) : \mathbb{R}^n \to \mathbb{R}$ is the objective function. We assume that there exists at least one global minimizer point in $X$, that is also a stationary point.

The considered algorithm is based on an inclusion function calculated by interval arithmetic or by other techniques.

**Properties of inclusion functions**

A function $F : \mathbb{I}^n \longrightarrow \mathbb{I}$ is an *inclusion function* of the objective function $f$ if for $\forall Y \in \mathbb{I}^n$ and $\forall y \in Y$ $f(y) \in F(Y)$, where $\mathbb{I}$ stands for the set of all closed real intervals.

$F$ is said to be an *isotone inclusion function* over $X$ if for $\forall Y, Z \in \mathbb{I}(X)$, $Y \subseteq Z$ implies $F(Y) \subseteq F(Z)$.

We call the inclusion function $F$ an *$\alpha$-convergent inclusion function* over $X$ if for $\forall Y \in \mathbb{I}(X)$ $w(F(Y)) - w(f(Y)) \leq C w^{\alpha}(Y)$ holds, where $\alpha$ and $C$ are positive constants.

We say that the inclusion function $F$ has the *zero convergence property*, if $w(F(Z_i)) \rightarrow 0$ holds for all the $\{Z_i\}$ interval sequences for which $Z_i \subseteq X$ for all $i = 1, 2, \ldots$ and $w(Z_i) \rightarrow 0$.

**Step 1** Let $L$ be an empty list, the leading box $A := X$, and the iteration counter $k := 1$. Set $\tilde{f} = \overline{F}(X)$.

**Step 2** Subdivide $A$ into $s$ subsets $A_i, (i = 1, \ldots, s)$ satisfying $A = \cup A_i$ so that $\text{int}(A_i) \cap \text{int}(A_j) = \emptyset$ for all $i \neq j$ where int denotes the interior of a set. Evaluate the inclusion function $F(X)$ for all the new subintervals, and update the upper bound $\tilde{f}$ of the global minimum.

**Step 3** Let $L := L \cup \{(A_i, \underline{F}(A_i))\}$.

**Step 4** Discard certain elements from $L$ that cannot contain a global minimum point.

**Step 5** Choose a new $A \in L$ and remove the related pair from the list.

**Step 6** While termination criteria do not hold let $k := k + 1$ and go to Step 2.

**Algorithm parameters**

The *generalized RejectIndex*:

$$pf(f_k, X) = \frac{f_k - \underline{F}(X)}{\overline{F}(X) - \underline{F}(X)}$$

is an algorithm parameter, the large value of which indicates that an interval $X$ is close to a minimizer point ($f_k \to f^*$, where $f^*$ is the global minimum).

The natural validated bounds on the $f_k$ values are:

$$\underline{f}_k = \min\{\underline{F}(Y^l), l = 1, \ldots, |L|\} \leq f_k \leq \tilde{f} = \overline{f}_k.$$

**Convergence properties 1**

THEOREM 1 [1]: Assume that the inclusion function of the objective function is isotone and it has the zero convergence property. Consider the interval branch-and-bound optimization algorithm that uses the cut-off test, the monotonicity test, the interval Newton step and the concavity test as accelerating devices, and that selects as next leading interval that interval $Y$ from the working list which has the maximal $p(f_k, Z)$ value.

A necessary and sufficient condition for the convergence of this algorithm to a set of global minimizer points is that the sequence $\{f_k\}$ converges to the global minimum value $f^*$ and there exist at most a finite number of $f_k$ values below $f^*$.

## Convergence properties 2

THEOREM 2 [2]: Assume that the inclusion function of the objective function is isotone and it has the zero convergence property. Consider the interval branch-and-bound optimization algorithm that uses the cut-off test, the monotonicity test, the interval Newton step and the concavity test as accelerating devices, and that selects as next leading interval that interval $Y$ from the working list which has the maximal $p(f_i, Z)$ value.

The algorithm converges exclusively to global minimizer points if

$$\underline{f}_k \leq f_k < \delta(\overline{f}_k - \underline{f}_k) + \underline{f}_k$$

holds for each iteration number $k$, where $0 < \delta < 1$.

**Proof**

Notice first that the maximal $pf(f_k, Y)$ values are always nonnegative, since $f_k$ is not less than the minimal lower bound of $F$. Due to $f_k < \tilde{f}$, the numerator of $pf$ is less than $\tilde{f} - \min\{\underline{F}(Y^l), l = 1, \ldots, |L|\}$. $\underline{f}_k$ is conservative, i.e. it is monotonously nondecreasing (based on the isotone inclusion functions). The same property is ensured for $\overline{f}_k$ by the isotonicity of $F(X)$, and by the updating of $\tilde{f}$. Thus $\underline{f}_k$ is monotonously nondecreasing, and $\overline{f}_k$ is monotonously nonincreasing.

Consider now an arbitrary point $x' \in X$ in such a way that $f(x') > f^*$, and that there is a subsequence $\{Y_{k_l}\}$ of the leading boxes that converges to $x'$. For this point $x'$ the sequence of lower bounds $\underline{F}(Y_{k_l})$ converges to $f(x')$ due to the zero convergence property, and obviously the sequence of upper bounds $\tilde{f}_k = \overline{f}_k$ on the minimum value converges to a value not greater than $f(x')$.

**Proof (continued / 2)**

In the same time the $f_k$ values must be below $f(x')$ from a certain iteration index $K$, since they fulfill the condition

$$\underline{f}_k \leq f_k < \delta(\overline{f}_k - \underline{f}_k) + \underline{f}_k$$

with a $0 < \delta < 1$. Then the respective $pf$ values are negative from an index $K' \geq K$.

If there are more such points as $x'$, then the above reasoning holds for each of them. In other words, also in this case from a certain index all $pf$ is negative.

**Proof (continued / 3)**

On the other hand, there is always at least one global minimizer point, a stationary point in one of the subintervals in the list $L$. The respective subinterval cannot be deleted by an accelerating step, and thus its $pf(f_k, Y)$ value is nonnegative. But this contradicts that a subinterval with a negative $pf$ value is selected, i.e. no subsequence of the generated intervals can converge to a nonoptimal point of the search region. □

**Numerical testing environment**

- The numerical tests were made on a Pentium-IV computer (1,4 Ghz, 1 Gbyte) under Linux.
- The inclusion functions were implemented via the PROFIL – BIAS routines. The programs were coded in C++.
- The basis algorithm was that of the C++ Toolbox for Verified Computing.
- The standard time unit was 0.00076 seconds.
- The new method assumed an approximate optimum value of 4 digits precision obtained by a previous traditional optimization algorithm.

**Numerical results, basic algorithm ($\epsilon = 0.01$)**

| Problem | | CPU time in seconds (Pentium IV, 1.4 Ghz) | | | | | | |
|---|---|---|---|---|---|---|---|---|
| name | $n$ | $\underline{F}$ | $(\overline{f}_k + \underline{f}_k)/2$ | % | new | % | $pf^*$ | % |
| H3 | 3 | 347.64* | 431.98* | 124 | 8.46 | 2 | 5.59 | 2 |
| H6 | 6 | 444.75* | 439.99* | 99 | 375.53* | 84 | 368.55* | 83 |
| GP | 2 | 474.79* | 1,760.60* | 371 | 3.09 | 1 | 3.48 | 1 |
| SHCB | 2 | 362.53* | 298.12 | 82 | 0.45 | 0 | 0.54 | 0 |
| L3 | 2 | 387.02* | 443.24* | 115 | 0.07 | 0 | 0.09 | 0 |
| L5 | 2 | 381.78* | 319.82* | 84 | 0.03 | 0 | 0.05 | 0 |
| Sch27 | 3 | 114.40 | 0.06 | 0 | 0.04 | 0 | 115.27 | 101 |
| EX2 | 5 | 358.43* | 354.16* | 99 | 311.11* | 87 | 328.91* | 92 |

* Unsolved due to the memory limitation (at most 20.000 intervals).

**References**

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