Consistency Techniques over Real Numbers

Laurent Granvilliers

granvilliers@irin.univ-nantes.fr

University of Nantes

Reduction and Propagation

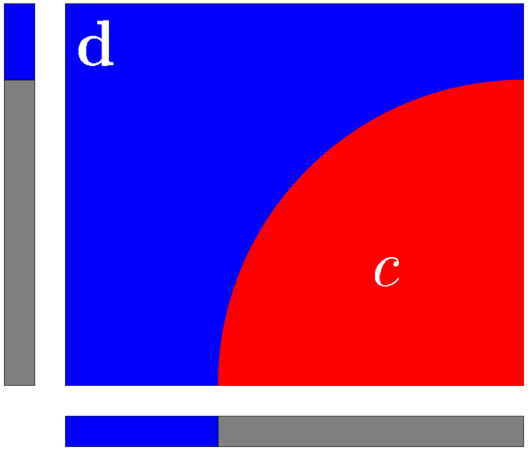
- Compute the solution set of $\{y = x^2, x \geqslant y + 1\}$ in the box $[-10, 10]^2$
- Propagation = sequence of reductions

$$
\begin{array}{ccc}
y = x^2 & \Longrightarrow & y \in [0, 10] \\
x \geqslant y + 1 & \Longrightarrow & x \in [1, 10] \\
y \leqslant x - 1 & \Longrightarrow & y \in [0, 9] \\
y = x^2 & \Longrightarrow & y \in [1, 9] \\
y = x^2 & \Longrightarrow & x \in [1, 3] \\
\ldots & \ldots & \ldots
\end{array}
$$

Projections of a Constraint

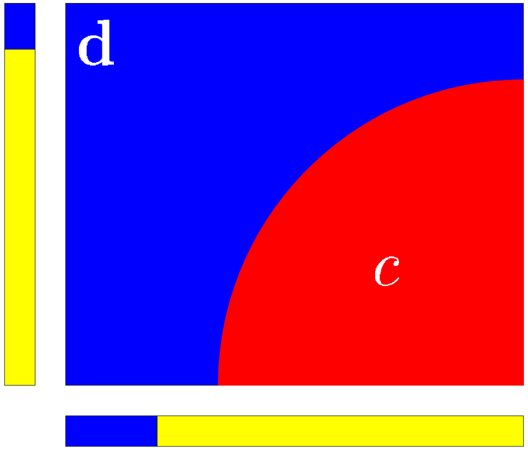
- Constraint $c(x_1, \ldots, x_n) \subseteq \mathbb{R}^n$
- Box $\mathbf{d} = \boldsymbol{d}_1 \times \cdots \times \boldsymbol{d}_n$
- Projection of c over x_i

$$\{a_i \in \boldsymbol{d}_i \mid \begin{array}{l} \exists a_1 \in \boldsymbol{d}_1, \ldots, \\ \exists a_{i-1} \in \boldsymbol{d}_{i-1}, \\ \exists a_{i+1} \in \boldsymbol{d}_{i+1}, \ldots, \\ \exists a_n \in \boldsymbol{d}_n : \\ c(a_1, \ldots, a_n)\} \end{array}$$



Approximation of Projections

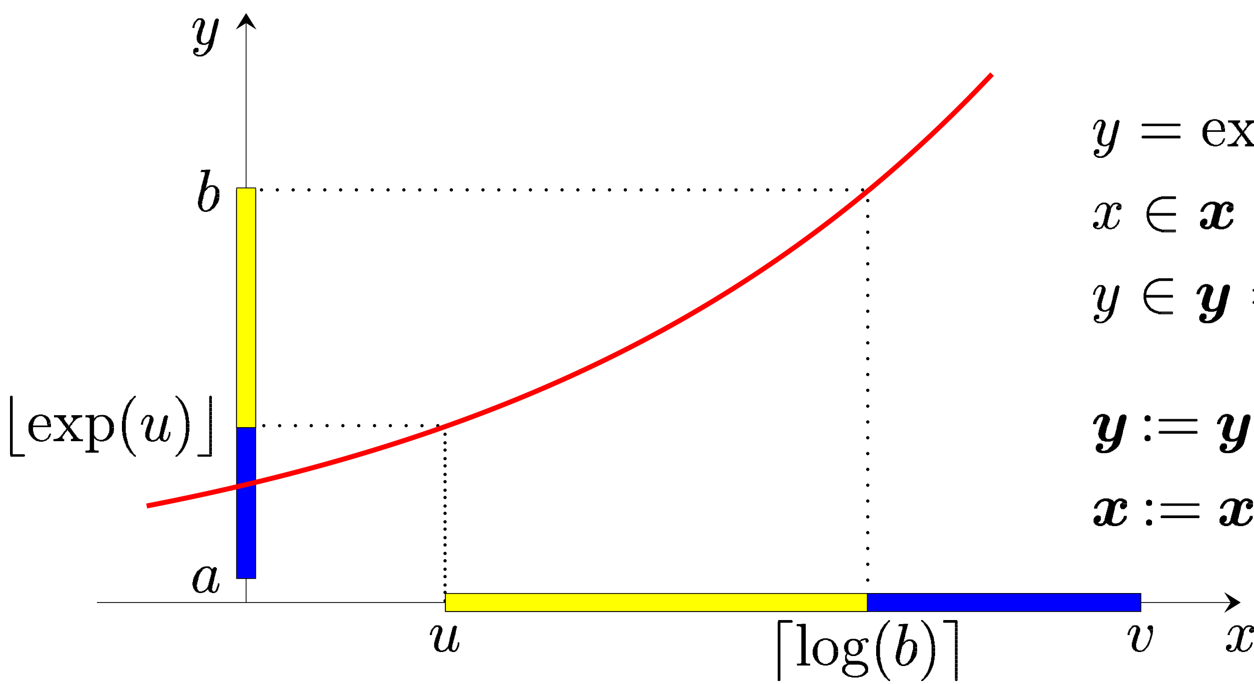
- Projections cannot be represented by floating-point intervals
- Approximation



- Reduction is computation of some yellow box

Reduction: Inversion

- Cleary 87
- Reduce the domains of x and y st. $y = \exp(x)$



$$y = \exp(x)$$
$$x \in \boldsymbol{x} = [u, v]$$
$$y \in \boldsymbol{y} = [a, b]$$

$$\boldsymbol{y} := \boldsymbol{y} \cap \exp([u, v])$$
$$\boldsymbol{x} := \boldsymbol{x} \cap \exp^{-1}([a, b])$$

Inversion for Complex Constraints

- Cleary 87
- Decomposition into primitives

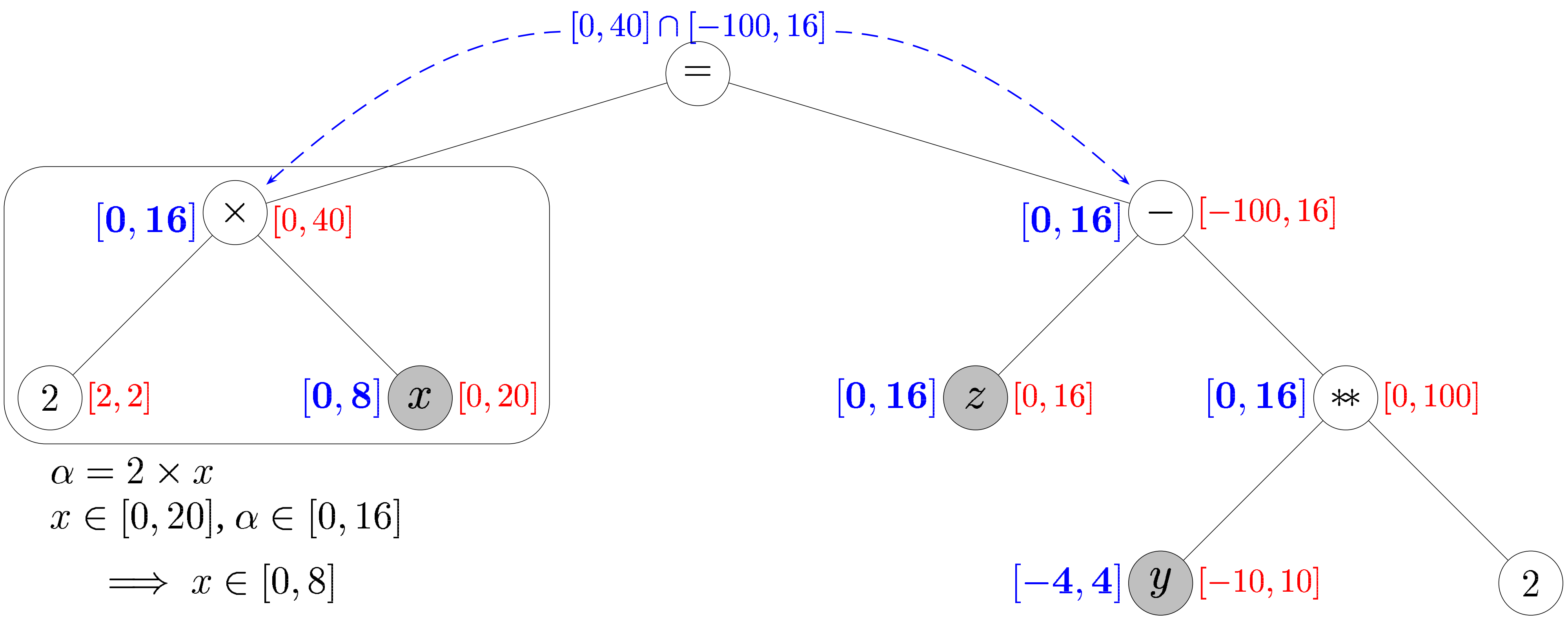
$$2x = z - y^2 \iff \begin{cases} \alpha & = & 2 \times x \\ \beta & = & y^2 \\ \alpha & = & z - \beta \end{cases}$$

for some $\alpha, \beta \in \mathbb{R}$

- Slow compilation and propagation processes

Inversion for Complex Constraints

- Benhamou *et al.* 99



Dependency Problem

- The inversion technique can be weak

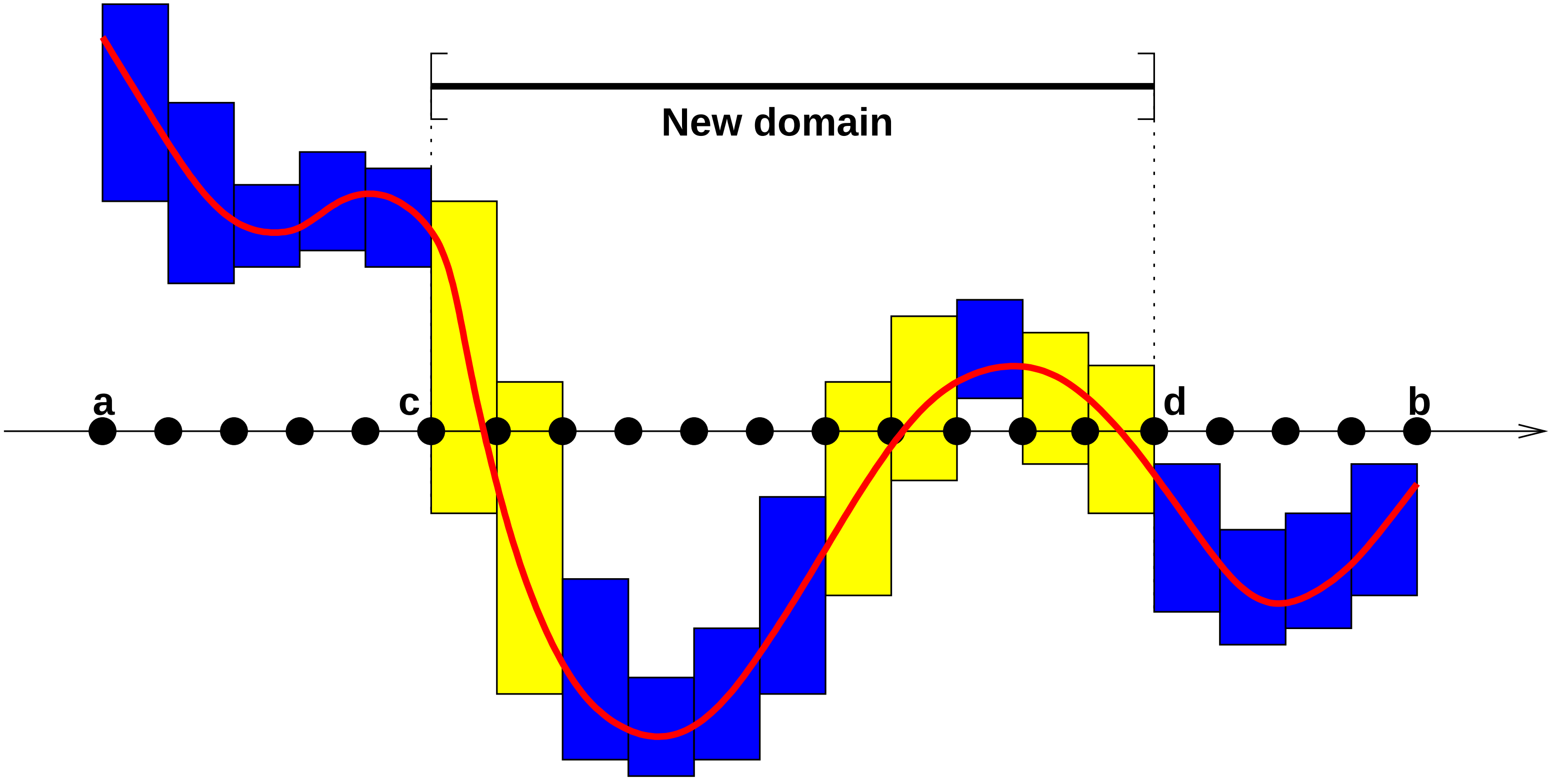
  $x + x = 0$, $x \in \boldsymbol{d} = [-1, 1]$

$$\begin{aligned} \boldsymbol{d} &:= \boldsymbol{d} \cap (-\boldsymbol{d}) \\ &:= [-1,1] \cap [-1,1] \\ &:= [-1,1] \end{aligned}$$

- Solutions
  - symbolic transformation
  - bisection-evaluation process

Reduction: Box-consistency

- Benhamou & McAllester & Van Hentenryck 94
- Reduce the domain of x st. $f(x) = 0$



Box-co. for Multivariate Constraints

- Consider a constraint $f(x_1, \ldots, x_n) = 0$ and an interval form $\boldsymbol{f}$ of f
- Given a box $\mathbf{d}$ box-consistency is computed for the set of interval functions

$$\begin{cases} \boldsymbol{f}_1(x_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \ldots, \boldsymbol{d}_n) \\ \boldsymbol{f}_2(\boldsymbol{d}_1, x_2, \boldsymbol{d}_3, \ldots, \boldsymbol{d}_n) \\ \qquad\qquad \ddots \\ \boldsymbol{f}_n(\boldsymbol{d}_1, \boldsymbol{d}_2, \ldots, \boldsymbol{d}_{n-1}, \boldsymbol{x}_n) \end{cases}$$

Reduction Model

- Let $\mathbb{I}$ be the set of closed intervals
- A reduction operator is a function $\mathbb{I}^n \to \mathbb{I}^n$ which is for all $x, y \in \mathbb{I}$
  - monotonic: $x \subseteq y \implies \theta(x) \subseteq \theta(y)$
  - contracting: $\theta(x) \subseteq x$
- Consistency techniques are monotonic and contracting
- These techniques are also verified!

Propagation Model

- Narin'yani 1983, Benhamou 1996, Apt 1998
- Lemma: given a finite set of reduction operators and a domain d the greatest common fixed-point of the operators included in d exists
- Propagation = application of operators
  - fair strategy $\Longrightarrow$ convergence
  - finiteness of domain $\Longrightarrow$ termination
  - independence wrt. the strategy

Fixed-point Propagation Algorithm

```
S := {O1,...,On}
Box d
repeat
   choose O in S
   save := d
   d := O(d)
   if d = save then
     S := S - {O}
   else
     S := {Oj | Oj depends on a modified domain}
until S is empty
```

Solver Cooperation

1. Generation of reduction operators
   - Inversion
   - Box-consistency
   - Interval methods
   - Simplex
2. Propagation over the set of operators
   - good strategy (choose) $\Longrightarrow$ efficiency

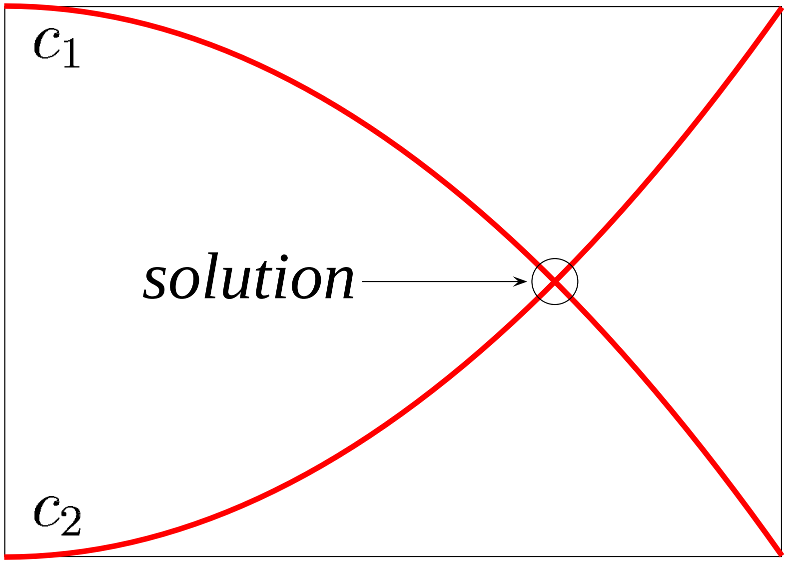
A Strategy

```
S := empty
for each constraint C do
  if there exists X in C occurring once
  then S := S U { inversion(c) }

  for each variable X in C do
    if X occurs more than once in C
    then S := S U { box(C,X) }
od
if a square system of equations E can be computed
then S := S U { IntervalNewton(E) }
```

Locality Problem

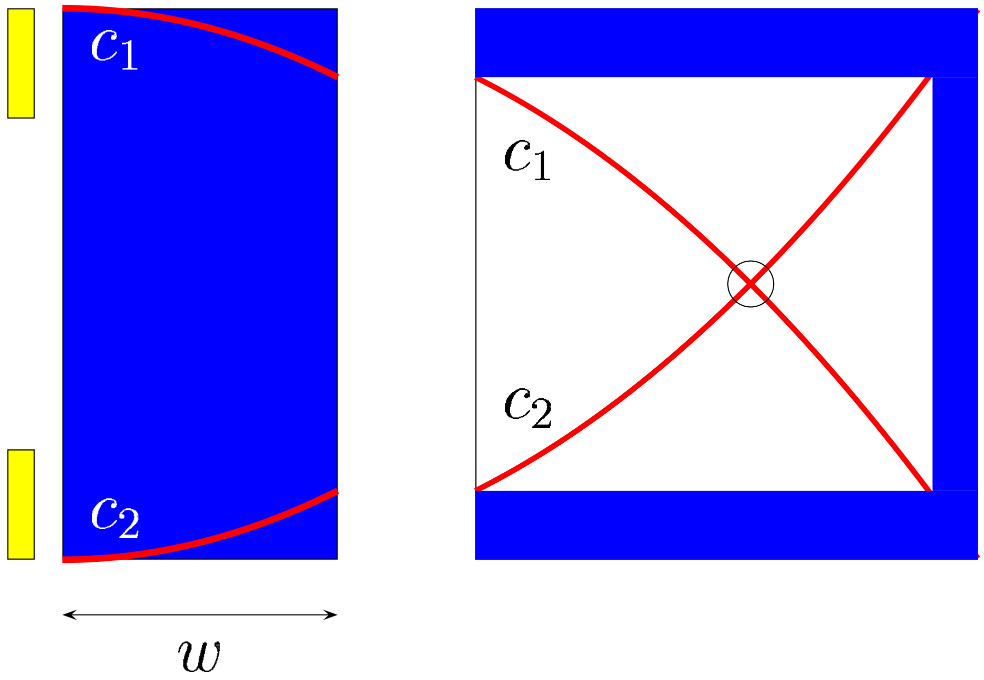
- Consider a difficult problem for inversion or box-consistency



- The intersection of projections is weaker than the projection of the intersection

Reduction: 3B-consistency

- Lhomme 1993
- Prove the inconsistency of a sub-domain at one bound of a variable domain



Bisection Algorithm

- Classical algorithm
  - precision of boxes: 10^{-8}
  - choice of variable: round-robin
  - bisection in 3 parts
- Reductions = cooperation of
  - I: inversion
  - B: box-consistency
  - N: interval Newton

Results

Problem	v	n	I-B-N	I-B	B-N	I-N
BB	10	1	0.1	0.2	0.1	3.4
	20	1	0.2	0.5	0.2	?
	40	1	0.5	1.0	0.5	?
	80	1	1.1	2.1	1.1	?
	160	1	2.2	4.1	2.2	?
MC 1	10	1	0.0	0.1	0.1	0.1
	20	1	0.1	0.4	0.7	0.4
	40	1	0.5	1.7	5.2	1.4
	80	1	2.4	6.9	41.4	6.4
Neuro	8	8	46.1	178.9	141.9	46.1
Kin 2	8	10	25.0	1175.2	53.3	19.2
Kin 1	12	16	0.7	110.9	2.0	0.7

Conclusion

- Consistency techniques are used in projects
  - automatic control (CNRS group on set-based methods)
  - computer-aided design (EPFL, RNTL CO2)
  - global optimization (COCONUT project)
  - image synthesis (PRIAMM project)
- Research directions: new types of constraints, specific propagation algorithms, inner approximations, quantified constraints, mixed-problems. . .