



Interval Computations in Maple using intpakX v1.0

Schloss Dagstuhl, 01/2003

Overview

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 - Types and operations
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 - Range Enclosure (2D and 3D)
 - Complex Disc Arithmetic

History

- 1993: `intpak` by R. Corless and A. Connell
 - Interval Type
 - Basic Arithmetic operations
 - Standard Functions
- 1999: `intpakX` by W. Krämer and I. Geulig (Add-on)
 - Interval Newton Method
 - Range Enclosure
 - Complex Disc Arithmetic
 - Graphical output

intpakX V1.0

- Update for Maple from Version 6
- Integrates *intpak* and *intpakX* (formerly separate packages)
- Redesign as Maple-Module
- Language integration
- Published as a *Maple PowerTool Interval Arithmetic* in 12/2002

Real Intervals

Interval type:

- Maple `List` with additional properties
(i.e. intervals $[x_1, x_2]$,
 x_1, x_2 float or \pm infinity, $x_1 \leq x_2$)
- `construct`, `convert` and `inapply` („interval unapply“)

Operators and standard functions

- Operators $\&+$, $\&-$, $\&*$, $\&/$ for intervals
 - Additional function for extended division
 - interval intersection/union
 - Power, square/root, logarithm and exponential functions
 - trigonometric functions
 - Rounding functions included
- Verified computations with guaranteed interval bounds possible (provided appropriate use of the Maple variable `Digits`)

Applicability

- suited for numerical computations as the operators described above or as the applications presented later

Effects to be paid attention to:

- Computations that need procedures other than the ones mentioned or that need to identify special operators; i.e. the **degree** function does not work with interval polynomials like

$$[1.0, 2.0] \&* x^2 \&+ [-1.5, -1.0]$$

- No automatic simplification of functions:

$$A \&+ 2 \&* A \text{ is not simplified into } 3 \&* A.$$

(can affect the success of symbolical computations).

Applications(I) Interval Newton Method

Iteration:

$$[x]^0 := [x], \quad [x]^{k+1} := \left(m([x]^k) - \frac{f(m([x]^k))}{f'([x]^k)} \right) \cap [x]^k$$

$m([x]^k)$ midpoint of $[x]^k$

- Computes enclosing intervals for *all* zeros of a cont. diff. function f with guaranteed bounds
- arbitrary start interval
- automatical use of interval operators/functions
- adjustable precision (digits, diameter of solution interval) and number of iterations
- graphical output of iteration steps (optional)

Applications(II) Range Enclosure

- for real-valued functions of one or two real variables
- in 2D, interval evaluation and mean value form are combined
- in 3D, only interval evaluation is done
- iterative interval subdivision
- automatical use of interval operators/functions
- adjustable number of iterations

Complex Intervals

- Disc intervals: `List` with three numerical values, interpreted as midpoint and radius (non-negative) of the disc
- Arithmetic operations: `&cadd`, `&csub`, `&cmult`, `&cdiv` (centered)
- Additional: Area-optimal multiplication and division
- Exponential function for complex discs

An Application:

Range Enclosure for Complex Polynomials

- Range Enclosure based on a Horner-Scheme using centered multiplication

`&horner_eval_cent`

- Range Enclosure based on a Horner-Scheme using area-optimal multiplication

`&horner_eval_opt`

- Range Enclosure using centered forms (similar to mean value form for real numbers)

`¢red_form_eval`

Example 1: Interval Newton Method

Enclosure of the zeros of

```
f := x -> sin(exp(sqrt(x - 2)));
```

on the interval [8.,10.]:

Load `intpakX`:

```
> restart;  
> libname:="/opt/lib/mymaplelib", libname;  
> with(intpakX);
```

Enclose zeros:

```
> f:=x->sin(exp(sqrt(x-2)));  
> X:=[8.,10.];  
> compute_all_zeros_with_plot(f,X,0.001);
```

$$f := x \rightarrow \sin(\exp(\sqrt{x-2}))$$

```
> compute_all_zeros_with_plot(f,X,0.001,10,10);
```

Digits = 10

Iteration step 1

xold= [8., 10.]

xnew1= [9.289288473, 10.]

xnew2= [8., 8.710711527]

Iteration step 2

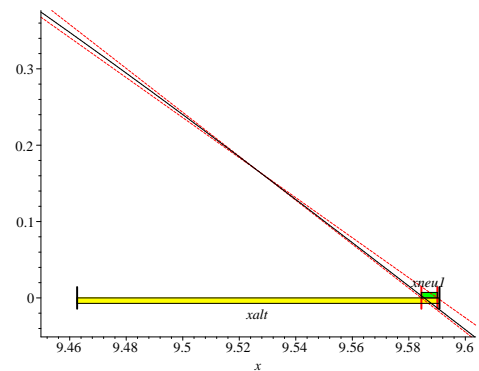
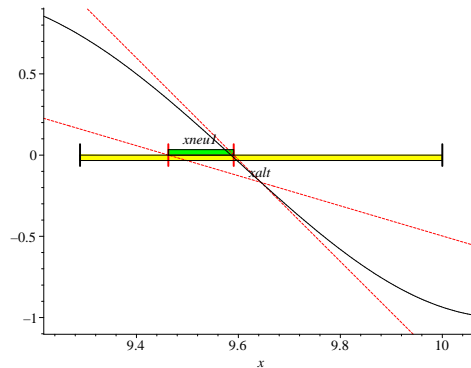
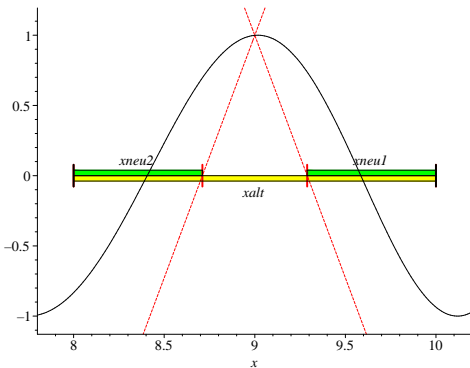
xold= [9.289288473, 10.]

xnew1= [9.462634907, 9.590834649]

Iteration step 3

xold= [9.462634907, 9.590834649]

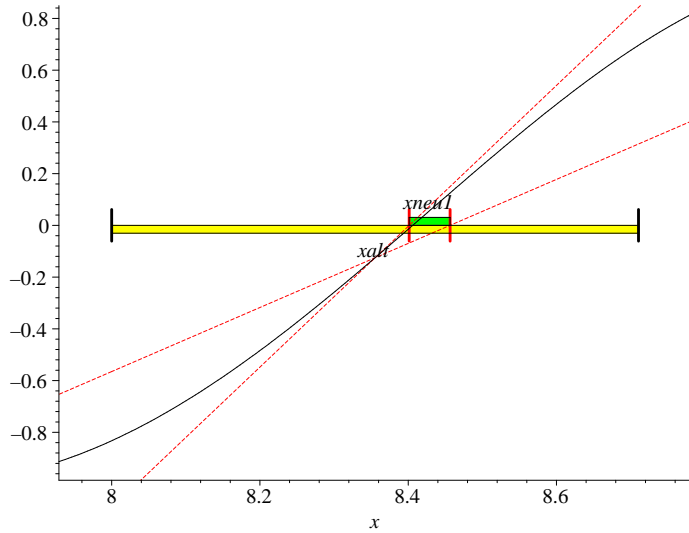
xnew1= [9.584440425, 9.590102305]



Iteration step 4

$x_{old} = [8., 8.710711527]$

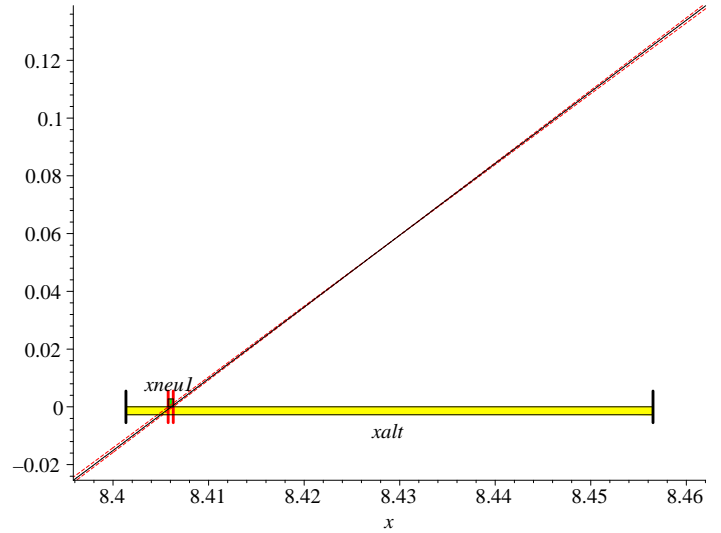
$x_{new1} = [8.401353571, 8.456507702]$



Iteration step 5

$x_{old} = [8.401353571, 8.456507702]$

$x_{new1} = [8.405771237, 8.406299401]$



Example 2: Range Enclosure (2D)

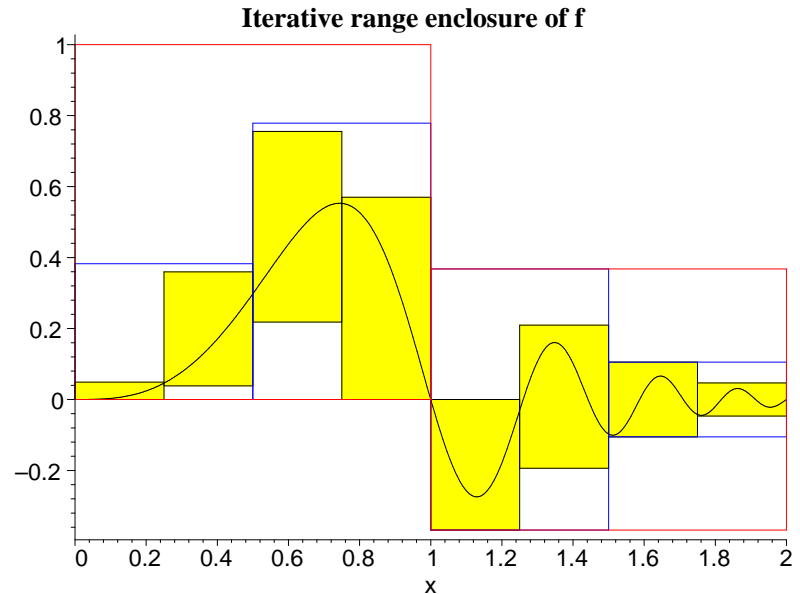
Range Enclosure of

```
f:=x->exp(-x^2)*sin(Pi*x^3);
```

```
> f:=x->exp(-x^2)*sin(Pi*x^3);  
> X:=[0.,2.];  
> compute_range(f,X,3);
```

Start range enclosure = [-1.000000002, 1.000000002]

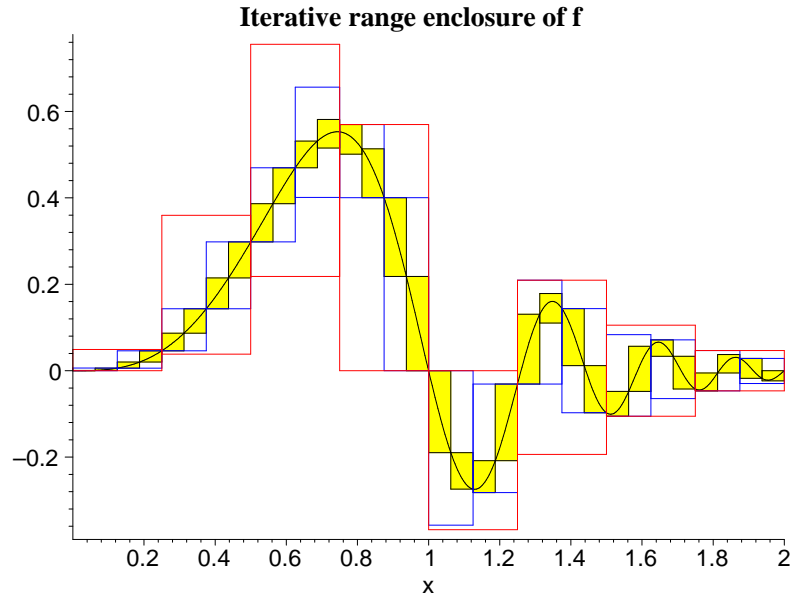
Range enclosure after step 3 = [-.3678794418,.7554611004]



```
> compute_range(f,X,5);
```

Start range enclosure = [-1.000000002, 1.000000002]

Range enclosure after step 5 = [-.2820629525,.5815483768]



Example 2: Range Enclosure (2D)

- Reasonable results displayed after few iteration steps
- Numerical values available in variables
`list_of_intervals` and `list_of_ranges`

```
> list_of_ranges;
```

```
[[.2980341562, .3644580713], [.3644580699, .4296256944], [.4296256931, .4877217319],  
 [.4877217307, .5314384278], [.5314384254, .5523340343], [.5348860212, .5680755514],  
 [.4922683804, .5417224422], [.4003403046, .4922683817], [.2688894620, .4003403067],  
 [.1101516556, .2688894645], [-.5312103680e-1, .1101516586], [-.1896402434, -.5312103191e-1],  
 [-.2663280673, -.1896402391], [-.2920842800, -.2460035260], [-.2599399733, -.1706423650],  
 [-.1706423681, -.3075637699e-1], [-.3075637973e-1, .1002647343], [.1002647315, .1599864556],  
 [.1197822643, .1643652721], [.1173607362e-1, .1197822664], [-.8297860010e-1, .1173607573e-1],  
 [-.1104304755, -.7204027092e-1], [-.9203610994e-1, -.1800481925e-1],  
 [-.1800482181e-1, .5649041403e-1], [.4840145700e-1, .7131668286e-1],  
 [-.1254422882e-1, .5228321860e-1], [-.5212547117e-1, -.1065394717e-1],  
 [-.4426684404e-1, -.5350768968e-2], [-.5350770488e-2, .3072094080e-1],  
 [.6012399421e-2, .3151551268e-1], [-.2177923802e-1, .6012400536e-2],  
 [-.2204438709e-1, .4408877421e-9]]
```

Example 3: Range Enclosure(3D)

```
g:=(x,y)->exp(-x*y)*sin(Pi*x^2*y^2);
```

Evaluation on the interval

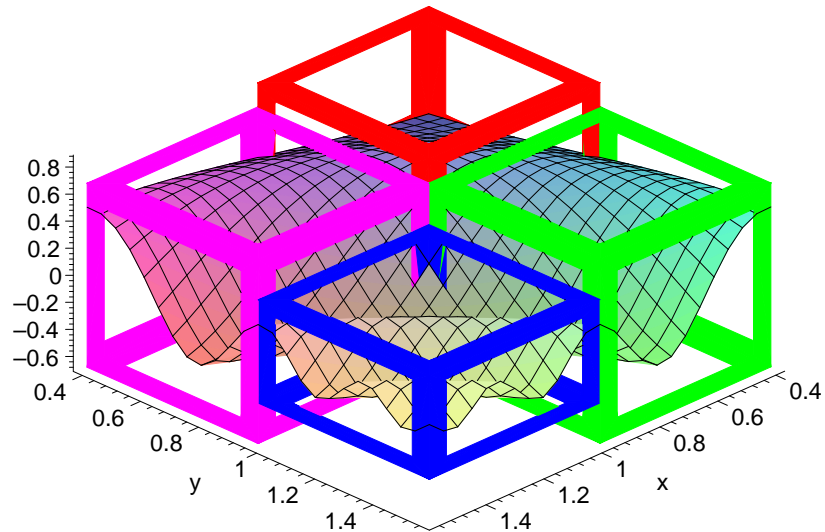
$[\frac{\pi}{8}, \frac{\pi}{2}] \times [\frac{\pi}{8}, \frac{\pi}{2}]$:

```
> compute_range3d(g,T,S,2);
```

Start range enclosure = [-.8570898115, .8570898115]

Range enclosure after step 1 = [-.8570898115, .8570898115]

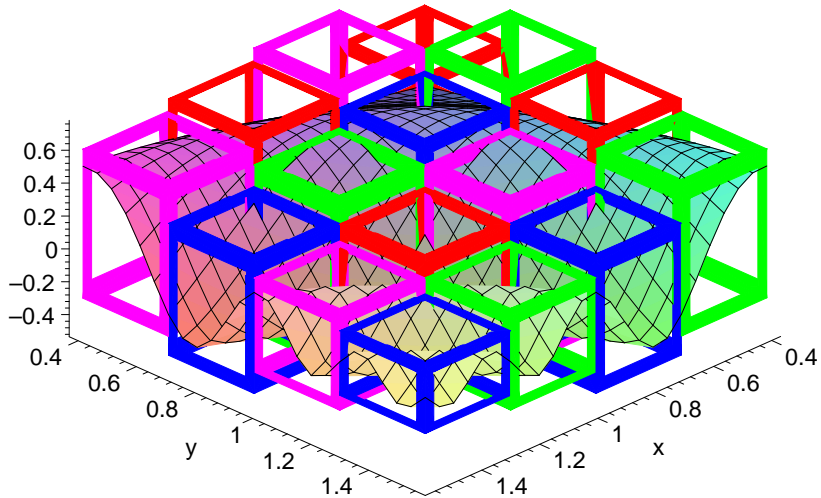
Range enclosure after step 2 = [-.6800891261, .8570898115]



$g := (x, y) \rightarrow \exp(-x * y) * \sin(\text{Pi} * x^2 * y^2);$

> compute_range3d(g,T,S,4);

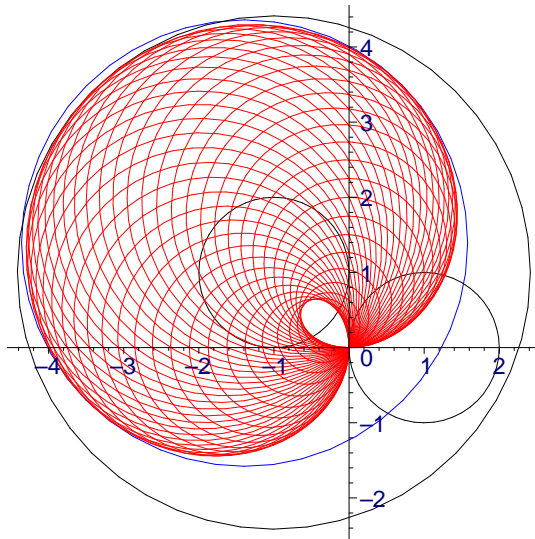
Start range enclosure = ' [-.8570898115, .8570898115]
Range enclosure after step 1 = [-.8570898115, .8570898115]
Range enclosure after step 2 = [-.6800891261, .8570898115]
Range enclosure after step 3 = [-.6800891261, .8486122905]
Range enclosure after step 4 = [-.5093193828, .7559256232]



Example 4: Complex Arithmetic

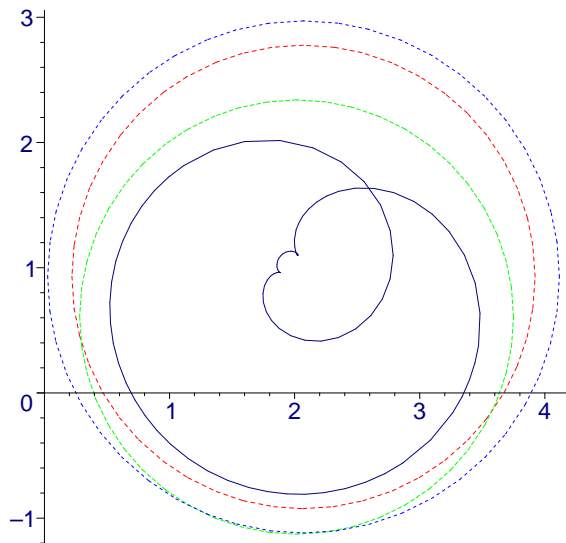
Multiplication of $\langle 1, 1 \rangle$ and $\langle -1+i, 1 \rangle$

- with centered multiplication
- with area-optimal multiplication



Range Enclosure for Complex Polynomials

$$p := (0.1 + 0.1i)z^5 + 0.2iz^4 - 0.1iz^3 + (-0.2 - 0.1i)z + 2.0 + 1.0i;$$



Results using Horner-Scheme with centered and area-optimal multiplication and centered form.

Tasks and Questions

- Adaptation to new concept of type definitions in Maple 8
- Output capabilities?
- Operator overloading?
- Symbolical computation capabilities?

Availability/Contact

Available from Waterloo Maple™ as

Maple PowerTool *Interval Arithmetic*

<http://www.mapleapps.com/powertools/interval/Interval.shtml>

or directly from our Resaerch Group's website at

<http://www.math.uni-wuppertal.de/wrswt/software/intpakX/>

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