On Some Algebraic Properties of Stochastic Numbers

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1 Introduction

Interval arithmetic and stochastic arithmetic have been both developed for the same purpose, i. e. to control errors coming from floating point arithmetic of computers and validate the results of numerical algorithms performed on computers. Interval arithmetic delivers guaranteed bounds for numerical results but requires special analysis and algorithms. On the other hand stochastic arithmetic is a model for the Cestac method which provides confidence intervals with known probability and can be easily implemented in existing numerical softwares. This work continues our study from [1] of the algebraic properties of stochastic arithmetic based on the comparison with interval arithmetic in midpoint-radius form, and on the algebraic structures that are induced by the operations on the two sets (stochastic numbers and intervals) cf. [7].

In the present paper following similar developements of interval arithmetic we introduce spaces analogous to quasilinear spaces [5, 6].

2 Stochastic Arithmetic

Stochastic arithmetic has been mainly studied in [3, 4, 9]. A *stochastic number* X is a gaussian random variable with a known mean value m and a known standard deviation σ and is denoted $X = (m, \sigma)$. The set of stochastic numbers is denoted as $S = \{(m, \sigma) \mid m \in \mathbb{R}, \sigma \in \mathbb{R}^+\}$. Stochastic arithmetic is in fact a theoretical model for the discrete stochastic arithmetic which is used in the Cestac method in which m and σ are computed using a Monte-Carlo technique consisting in performing each arithmetic operation several times with an arithmetic with a random rounding mode, see [2, 8, 9]. Hence the Cestac method takes naturally into account the correlation between errors whereas stochastic arithmetic actually does not. Anyhow in most applications the results predicted with stochastic arithmetic are identical or very close to those provided by the Cestac method. Thus stochastic arithmetic is considered as giving a good algebraic model for the Cestac method which uses the following classical property.

Property: If $X = (m, \sigma) \in S$, $0 \le \beta \le 1$ and r is a realization of X, then there exist λ_β only depending on β, such that

$$P\left(r \in [m - \lambda_\beta \sigma, m + \lambda_\beta \sigma]\right) = 1 - \beta. \tag{1}$$

$I_{\beta,X} = [m - \lambda_\beta\ \sigma, m + \lambda_\beta\ \sigma]$ is the *confidence interval* of X with probability $1 - \beta$. Equality (1) is a well-known property of gaussian random variables. For $\beta = 0.05, \lambda_\beta \approx 1.96$. The Cestac method computes m and σ by sampling, stochastic arithmetic computes m and σ algebraically.

3 Arithmetic Operations Between Stochastic Numbers

Let $X_1 = (m_1, \sigma_1)$ and $X_2 = (m_2, \sigma_2)$ be two stochastic numbers. (Usual) equality between two stochastic numbers X_1, X_2 is defined by: $X_1 = X_2$, if $m_1 = m_2$ and $\sigma_1 = \sigma_2$.

In this work we concentrate on the operations addition

$$X_1 + X_2 = (m_1 + m_2, \sqrt{\sigma_1^2 + \sigma_2^2})$$

and multiplication by scalars $\gamma \in \mathbb{R}$

$$\gamma * X = (\gamma m, |\gamma|\sigma).$$

We have shown in [1] that the set S is an abelian monoid with respect to addition with cancellation law.

Multiplication by scalars satisfies:

a) *First distributive law:* $\lambda * (X + Y) = \lambda * X + \lambda * Y$;

b) *Associativity:* $\lambda * (\mu * X) = (\lambda\mu) * X$;

c) *Identity:* $1 * X = X$.

Remark. The second distributive law: $(\lambda + \mu) * X = \lambda * X + \mu * X$ does not hold in general. Moreover, it does not generally hold even for λ, μ nonnegative. We thus have no quasi-distributive law (as in the case of intervals).

The mean values satisfy the distributive law and thus form a linear space. The standard deviations satisfy the following law:

$$(\sqrt{\lambda^2+\mu^2}) * \sigma = \lambda * \sigma + \mu * \sigma, \quad \lambda \geq 0, \mu \geq 0,$$

or, equivalently,

$$(\sqrt{\lambda+\mu}) * \sigma = \sqrt{\lambda} * \sigma + \sqrt{\mu} * \sigma, \quad \lambda \geq 0, \mu \geq 0.$$

We investigate the space of standard deviations by embedding it in an additive group, obtaining thus a space close to a quasilinear space with group structure [6].

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