Computing Tight Bounds for the L_1-Norm of Peano Kernels

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The purpose of this paper is to point out essential consideration for computing tight bounds for the error constants required in the error estimation of a numerical quadrature rule.

Due to Peano, cf. [2, 6], the quadrature error $E(f) := I(f) - S(f)$ resulting from approximation to the definite integral $I(f) := \int_a^b w(x)\, f(x) dx$ by a quadrature rule $S(f) := \sum_{i=1}^{M} w_i\, f(x_i)$ of degree d, where $w_i > 0$, can be represented as

$$E(f) = \int_a^b K_r(t)\, f^{(r)}(t)\, dt\ , \qquad (1)$$

where $f \in C^r[a,b]$, $1 \le r \le d+1$ and the Peano kernel $K_r(t)$ is defined by

$$K_r(t) \ :=\ E_x\left((x-t)_+^{\langle r-1\rangle}\right)$$

with $(x-t)^{\langle r\rangle} := \dfrac{(x-t)^r}{r!}$ and

$$(x-t)_+^r := \begin{cases} (x-t)^r\ , & \text{for } \ x \ge t\ , \\ 0\ , & \text{for } \ x < t\ . \end{cases}$$

Generally, the quadrature error (1) is estimated according to

$$|E(f)| \le \|f^{(r)}\|_\infty \cdot \int_a^b |K_r(t)|\, dt\ , \qquad (2)$$

or for validated computation

$$E(f) \ \ \in \ \ f^{(r)}([a,b]) \cdot \int_a^b K_r^+(t)\, dt \ - \ f^{(r)}([a,b]) \cdot \int_a^b K_r^-(t)\, dt\ , \qquad (3)$$

where $K^+(t) := \max(K(t),\, 0)$, $K^-(t) := \max(-K(t),\, 0)$.

If $K_r(t)$ is definite on $[a, b]$, i. e. $K_r(t)$ does not change sign on $[a, b]$, then in (2), (3) there remains the integral $\int_a^b K_r(t)\, dt$ to be computed, which might not be a difficult task, at least not for $w(x) \equiv 1$. However, this is not the usual case. According to (1), we have

$$E(x^r) \;=\; r! \cdot \int_a^b K_r(t)\, dt \;, \qquad 1 \le r \le d+1 \;. \tag{4}$$

It follows immediately that $\int_a^b K_r(t)\, dt = 0$ for $1 \le r \le d$. This implies that only the Peano kernels of highest orders may be definite. If $K_{d+1}(t)$ is definite on $[a, b]$, then $\int_a^b K_r(t)\, dt$ is equal to $E(x^{d+1})/(d+1)!$, which can be easily computed. On the other hand, if $K_{d+1}(t)$ changes its sign on $[a, b]$ and/or f is not sufficiently smooth on $[a, b]$, i. e. $f \in C^r[a, b]$, $1 \le r \le d+1$, then for the error estimation (2) and (3) the error constants $\int_a^b K_r^+(t)\, dt$ and $\int_a^b K_r^-(t)\, dt$ have to be known, where $1 \le r \le d+1$.

A straightforward method for computing the error constant $\int_a^b |K_r(t)|\, dt$ is to compute all the zeros of $K_r(t)$ at first, then to integrate $K_r(t)\Big|_{[x_i, x_{i+1}]}$ between every two adjacent zeros within each subinterval $[x_i, x_{i+1}]$, cf. [11, 9, 10]. Since it is not easy to identify all the zeros of Peano kernels numerically, hence, in [11] the method was only applied to K_r for $r = 1, 2$, in [9] the method was only applied to K_d by analytically confirming that for Gauss-Legendre rules K_d possesses only the zero 0. In [10] the first trial for the whole range $1 \le r \le d+1$ was undertaken and interval computations was used. However, the numerical results presented in [11, 9, 10] reveal themselves to be validated or improved. In the literature there was also much effort given for estimating an upper bound for $\int_a^b |K_r(t)|\, dt$, cf. [2, 5, 3, 4]. Among them, good results in general can only be obtained for $r = d+1$. For $1 \le r \le d$, the most suggested upper bounds are relatively coarse. This paper adopts the same method used in [11, 9, 10] and has successfully gained significant improvement in the computational quality for the whole range $1 \le r \le d+1$.

In the presentation, essential consideration for doing the computation, the algorithms as well as numerical results with comparison to published bounds are proposed. All the ideas presented in this paper can also be applied to one-dimensional Sard kernels appearing in the error representation of a numerical cubature rule.

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