# Towards Validated Global Optimal Control

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Consider a discrete-time optimal control problem in the following *direct* form: choose control values $u_i \in R^p$ for each timestep $0 \leq i < N$ so as to minimize $z = F(x_N)$ where $x_0$ is some fixed constant and the state equation is $x_{i+1} = f_i(x_i, u_i)$ for $0 \leq i < N$. Here each $f_i$ is a smooth map from $R^q \times R^p \to R^q$ and $F$ is a smooth map from $R^q$ to $R$. The dimension of $u_i$ may depend upon the timestep $i$, but for notational convenience we omit this refinement.

We lose nothing by restricting attention to target functions of this form: the more usual formulation where $z$ has the form $z = \sum_{i=0}^{N-1} F_i(x_i, u_i) + F_N(x_N)$, can be reduced to the form $z = F(x_N)$ by augmenting each state $x_i$ with a new component $v_i \in R$ defined by $v_0 = 0; v_{i+1} = v_i + F_i(x_i, u_i)$ and defining $F(x_N, v_N) = v_N + F_N(x_N)$.

In the direct formulation the $Np$ independent variables are the controls $u_i : 0 \leq i < N$. Typically the number of timesteps, and hence the number of independent variables is very large (millions). This makes the validated solution of such problems difficult.

In the alternative *indirect* formulation, the only independent variables are the $q$ components of an initial *costate* $\tilde{x}_0$. At each timestep $i$, the current controls $u_i$ and the successor costate $\tilde{x}_{i+1}$ are implicitly defined, in terms of the current state $x_i$ and current costate $\tilde{x}_i$, by the costate equations and the Pontryagin equations

$$\tilde{x}_i - [f'_{x,i}(x_i, u_i)]^T \tilde{x}_{i+1} = 0; \quad [f'_{u,i}(x_i, u_i)]^T \tilde{x}_{i+1} = 0.$$

The state equation $x_{i+1} = f_i(x_i, u_i)$ then gives $x_{i+1}$ in terms of $x_i$ and $u_i$. In the indirect formulation, the requirement for the path to be optimal is that $\tilde{x}_N - F'(x_N) = 0$ which we observe can be regarded as a form of the transversality condition. For an optimal path (although not in general) the numerical values of the costates $\tilde{x}_i$ are equal to those of the *adjoint* states $\bar{x}_i = \partial z / \partial x_i$.

For a non-optimal path, the residual value $r = \tilde{x}_N - F'(x_N)$ of the transversality equation gives a measure of how far the initial costate value $\tilde{x}_0$ differs from that for the optimum path. An important advantage of the indirect approach for validated methods is the drastic reduction in the number of independent variables, from $Np$ to $q$.

In 1983 Pantoja described a computationally efficient stagewise construction of the Newton direction for the direct formulation. Recently [6] we formulated an indirect analogue of Pantoja's algorithm, which gives exactly the Newton step $a_0$ for the initial costate with respect to a terminal transversality condition. We believe this indirect reformulation of Pantoja's algorithm potentially forms a useful tool for attacking the problem of verified global optimal control using interval methods.

We conclude this abstract by giving a scalar (non-interval) form of the indirect Pantoja algorithm, and then indicate some of the possibilities.

**Step 1.** Given the fixed initial value for $x_0$, set a trial initial value for $\tilde{x}_0$. For $i$ from 0 up to $N-1$ calculate $u_i \in R^p; \tilde{x}_{i+1}, x_{i+1} \in R^q$ by solving the implicit costate and Pontryagin equations, respectively

$$\tilde{x}_i - \left[f'_{x,i}\right]^T \tilde{x}_{i+1} = 0; \qquad \tilde{u}_i = \left[f'_{u,i}\right]^T \tilde{x}_{i+1} = 0,$$

for $u_i$ and $\tilde{x}_{i+1}$ and setting $x_{i+1} = f_i(x_i, u_i)$.

**Step 2.** Set $z = F(x_N)$, and define $a_N \in R^q, D_N \in R^{q \times q}$ by

$$D_N = F''(x_N); \qquad a_N = -r \quad \text{where } r = \tilde{x}_N - F'(x_N).$$

**Step 3.** For $i$ from $N-1$ down to 0 calculate $a_i \in R^q; A_i, D_i \in R^{q \times q}; B_i \in R^{p \times q}; C_i \in R^{p \times p}$ by

$$A_i = \left[f'_{x,i}\right]^T D_{i+1} \left[f'_{x,i}\right] + (\tilde{x}_{i+1})^T \left[f''_{xx,i}\right]$$

$$B_i = \left[f'_{u,i}\right]^T D_{i+1} \left[f'_{x,i}\right] + (\tilde{x}_{i+1})^T \left[f''_{ux,i}\right]$$

$$C_i = \left[f'_{u,i}\right]^T D_{i+1} \left[f'_{u,i}\right] + (\tilde{x}_{i+1})^T \left[f''_{uu,i}\right]$$

where $[.]$ denotes evaluation at $(x_i, u_i)$, and we write (for example)

$$\left(\left[f'_{u,i}\right]^T D_{i+1} \left[f'_{x,i}\right]\right)_{j,k} \text{ for } \sum_{l=1}^{q} \sum_{m=1}^{q} \left[\frac{\partial (x_{i+1})_l}{\partial (u_i)_j}\right] (D_{i+1})_{l,m} \left[\frac{\partial (x_{i+1})_m}{\partial (x_i)_k}\right] \text{ etc.}$$

If $C_i$ is singular then the algorithm fails, otherwise set

$$D_i = A_i - B_i^T C_i^{-1} B_i$$

$$a_i = \left[f'_{x,i}\right]^T a_{i+1} - B_i^T C_i^{-1} \left[f'_{u,i}\right]^T a_{i+1}$$

and **STOP**.

Either the algorithm fails to terminate, or else at the end $a_0$ satisfies

$$\tilde{x}_N - F'(x_N) + a_0 \cdot \frac{\partial}{\partial \tilde{x}_0} (\tilde{x}_N - F'(x_N)) = 0.$$

Since in the region of an optimum path we have that all the $C_i$ are positive definite [6], the indirect algorithm can be combined with a variational analysis to provide a largeish box around the (believed) global optimum for $\tilde{x}_0$ in which interval Newton establishes that only one solution to the transversality equation exists.

For example if we set $r$ to be a cartesian basis vector, then the algorithm gives the corresponding row of $A = J^{-1}$, where $J = \partial r / \partial \tilde{x}_0$ for the midpoint of the box $[\tilde{x}_0] = \tilde{x}_0 + \Delta$. Using Automatic Differentiation techniques [1] we can differentiate through the costate and Pontryagin equations to evaluate ranges for the derivatives $B = [dr/d\tilde{x}_0]$. Then any optimal point in $[\tilde{x}_0]$ is also in $\phi([\tilde{x}_0] = \tilde{x}_0 + a_0 + \Delta[I - AB]$.

This should significantly ease the task of proving other boxes to either contain no solution to the transversality equations, or to be suboptimal.

We stress that the outline given here is very simplistic (it is assumed that all state and control constraints have been incorporated into the target function by penalty terms, for instance) and that much work remains to be done before global optima for control problems can be validated rigorously in a reasonable time. Nevertheless we believe that the approach set out here is a viable manifesto for a programme to achieve this.

# References


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