Convex–Concave Extensions for Polynomials

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A frequently used approach for solving nonlinear systems, combinatorial optimization, or constrained global optimization problems is the generation of relaxations and their use in a branch and bound framework. Generally speaking, a relaxation of a given problem has the properties that

(i) each feasible point of the given problem is feasible for the relaxation,

(ii) the relaxation is easier to solve than the given problem, and

(iii) the solutions of the relaxation converge to the solutions of the original problem, provided the maximal width of the set of feasible points converges to zero.

For many problems a relaxation can be constructed, if the functions which define the problem can be bounded from below by affine or, more generally, by convex functions.

In our talk we address the construction of convex lower bounding and equally concave upper bounding functions for multivariate polynomials. Both functions together constitute a so-called *convex–concave extension*. For polynomials this is obtained in a natural way if we represent the given polynomial (for simplicity we consider here only the univariate case and concentrate on the unit interval $I = [0,1]$)

$$p(x) = \sum_{i=0}^{n} a_i x^i$$

in its Bernstein form

$$p(x) = \sum_{i=0}^{n} b_i B_i(x)$$

where the

$$B_i(x) = \binom{n}{i} x^i (1-x)^{n-i},\ i = 0, 1, \ldots, n$$

are the Bernstein polynomials. The coefficients of this expansion, the so-called Bernstein coefficients, can easily be computed from the coefficients of p:

$$b_i = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{n}{j}} a_j,\ i = 0, 1, \ldots, n \quad (\text{note that } b_0 = p(0), b_n = p(1)).$$

A fundamental property of the Bernstein expansion is its *convex hull property*

$$\left\{ \begin{pmatrix} x \\ p(x) \end{pmatrix} : x \in I \right\} \subseteq conv \left\{ \begin{pmatrix} \frac{i}{n} \\ b_i \end{pmatrix} : i = 0, 1, \ldots, n \right\}$$

which states that the graph of p over I is contained in the convex hull (denoted by *conv*) of its control points. Based on this property, convex–concave extensions of increasing complexity can be constructed (we are giving here only the construction of the lower bounding function). E.g., we obtain an affine lower bounding function if we consider the straight line which passes through a facet of the lower part of the convex hull of the control points, the slope of which is given by the absolute value of the slope between the control points associated with the smallest and next to smallest Bernstein coefficients. A convex lower bounding function is provided by the lower part of the convex hull of the control points.

In Figures 1 and 2 convex–concave extensions for a polynomial of fourth degree over the intervals $[0, 0.5]$, $[0, 0.6]$, $[0, 0.7]$ and $[0, 1]$ are displayed. In Fig. 1 the extension is based on one affine upper and lower bounding function. The figure shows that this convex–concave extension is not inclusion isotone. In Fig. 2 the extension is provided by the convex hull of the control points. In this special example the convex hull is inclusion isotone. We show that this property holds generally. However, it should be noted that inclusion isotonicity is not a necessary prerequisite for constructing and using convex–concave extensions.

In the multivariate case the affine lower bounding function c can be characterized as the optimal solution of a linear programming problem. We present an upper bound for the difference $p - c$ which exhibits in the univariate case quadratic convergence with respect to the width of the interval.

Due to rounding errors, inaccuracies may be introduced into the calculation of the Bernstein coefficients and therefore of the bounding functions. This may lead to erroneous results in applications. We are giving some suggestions for the way in which the calculations have to be performed so that verified results are obtained.

Fig. 1. Failure of inclusion isotonicity with one affine function.

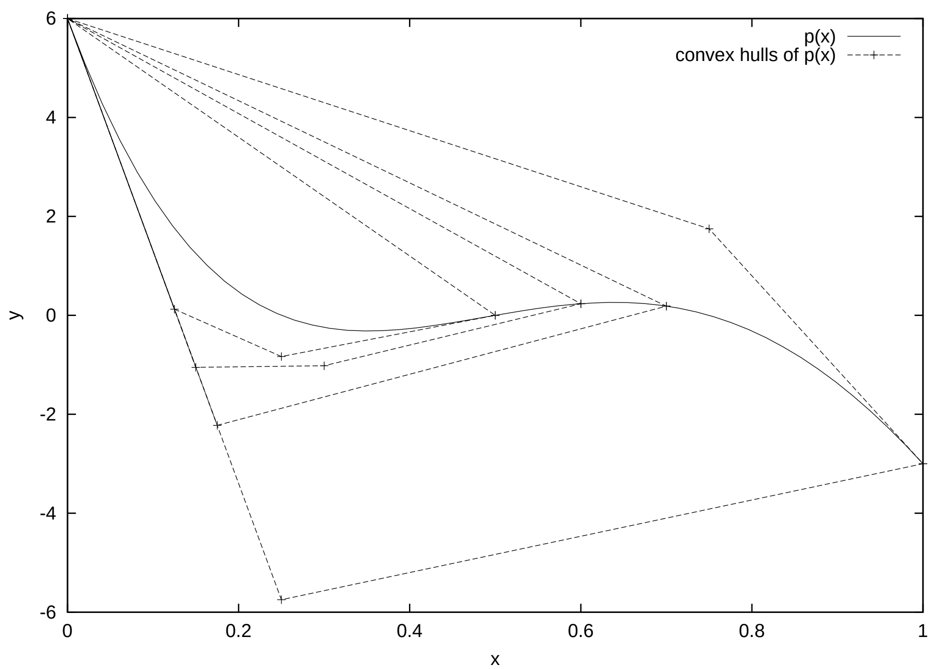

Fig. 2. The convex hull is inclusion isotone.