On Existence and Uniqueness Verification for Non-Smooth Functions

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1 Summary

It is known that interval Newton methods can verify existence and uniqueness of solutions of a nonlinear system of equations near points where the Jacobi matrix of the system is not ill-conditioned. Recently, we have shown how to verify existence and uniqueness, up to multiplicity, for solutions at which the Jacobi matrix is singular. We do this by efficient computation of the topological index over a small box containing the approximate solution. Algorithmically, our techniques mimic the non-singular case (both in algorithmic steps and computational complexity), and can be considered as *incomplete Gauss–Seidel sweeps*.

Since the topological index is defined and computable when the Jacobi matrix is not even defined at the solution, one may speculate that efficient algorithms can be devised for verification in this case, too. In this talk, we discuss, through examples, key techniques underlying our simplification of the calculations that cannot necessarily be used when the function is non-smooth. We also suggest when degree computations involving non-smooth functions may be practical.

Our examples also shed light on our published work on verification involving the topological degree.

2 Introduction and Some Details

Given a system of nonlinear equations $F(x) = 0$, numerical methods produce an approximation $\check{x}$ to a solution x^*. It is then sometimes desirable to compute bounds

$$\begin{aligned} \boldsymbol{x} &= (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n) \\ &= ([\underline{x}_1, \overline{x}_1], [\underline{x}_2, \overline{x}_2], \ldots, [\underline{x}_n, \overline{x}_n], \end{aligned}$$

such that $\check{x}$ is the center of $\boldsymbol{x}$, and such that $\boldsymbol{x}$ is guaranteed to contain a solution x^* to $F(x) = 0$. This leads to the problem

> Given $F : \boldsymbol{x} \to \mathbb{R}^n$, where $\boldsymbol{x} \in \mathbb{IR}^n$, *rigorously* verify:
>
> - there exists a $x^* \in \boldsymbol{x}$ such that $F(x^*) = 0$.

(1)

Here, $\mathbb{IR}^n$ represents the set of n-dimensional vectors, as $\boldsymbol{x}$, whose components are intervals.

If the Jacobi matrix $F'(x^*)$ is non-singular and continuous in $\boldsymbol{x}$, then we can use interval Newton methods to verify existence and uniqueness of $x^* \in \boldsymbol{x}$, $F(x^*) = 0$; see [3, Chapter 8], [4, pp. 219–223], and the references therein. Such interval Newton methods are of the form

$$\tilde{\boldsymbol{x}} = \boldsymbol{N}(F; \boldsymbol{x}, \check{x}) = \check{x} + \boldsymbol{v}, \tag{2}$$

where

$$\Sigma(\boldsymbol{A}, -F(\check{x})) \subset \boldsymbol{v}, \tag{3}$$

where $\boldsymbol{A}$ is a Lipschitz matrix for F over $\boldsymbol{x}$, and where

$$\Sigma(\boldsymbol{A}, -F(\check{x})) = \{x \in \mathbb{R}^n \mid \exists A \in \boldsymbol{A} \text{ with } AX = -F(\check{x})\}. \tag{4}$$

Here $\check{x}$ is some point in $\boldsymbol{x}$ (often taken to be its midpoint) that, in the context of this paper, we consider to be an approximate solution.

Theorem 1 *([4, Theorem 1.19, p. 62], originally from [8]) Suppose $\tilde{\boldsymbol{x}} = \boldsymbol{N}(F; \boldsymbol{x}, \check{x})$ is the image of $\boldsymbol{x}$ and $\check{x}$ under an interval Newton method. If $\tilde{\boldsymbol{x}} \subseteq \boldsymbol{x}$, it follows that there exists a unique solution of $F(x) = 0$ within $\boldsymbol{x}$.*

Recently, we have developed techniques that can verify existence of solutions to $F(x) = 0$ within $\boldsymbol{x}$, even when $F'(x) = 0$ for some $x \in \boldsymbol{x}$. These techniques are based on computing the *topological degree* $\mathrm{d}(F, \boldsymbol{x}, 0)$ of F over $\boldsymbol{x}$. If every $x \in \boldsymbol{x}$ where $F(x) = 0$ has the Jacobi matrix $F'(x)$ nonsingular, then $\mathrm{d}(F, \boldsymbol{x}, 0)$ is equal to the number of solutions of $F(x) = 0$ in $\boldsymbol{x}$ at which the determinant of $F'(x)$ is positive, minus the number of solutions of $F(x) = 0$ in $\boldsymbol{x}$ at which the determinant is negative. However, the integer $\mathrm{d}(F, \boldsymbol{x}, 0)$ is both continuous in F and depends only on values of F on the boundary $\boldsymbol{\partial x}$. Thus, in theory, F' may be singular, and indeed, even non-smooth, in the interior $\mathrm{int}(\boldsymbol{x})$.

Our recent work, as other work dealing with the topological degree, depends on a basic formula that relates the topological degree to solutions of a derived system over the boundary of $\boldsymbol{x}$. In contrast to previous literature on computing the topological degree, in our recent work in [1], [7], [5], [6], and [2], we are not given a large box $\boldsymbol{x}$, but we construct $\boldsymbol{x}$ sufficiently small to allow us to use a local model of F to both reduce the dimension of the search on the boundary and to greatly speed the resulting low-dimensional search. The process includes

1. preconditioning the system,

2. applying a local model to the preconditioned system to reduce the dimension, and

3. using a local model to predict where the solutions to the derived system are on $\boldsymbol{\partial x}$.

Our analysis indicates the verification proceeds in $\mathcal{O}\left(n^3\right)$ time for rank-defect 1 Jacobi matrices; this order has been verified experimentally with solutions to finite discretizations of a model problem with n up to 320.

The question we ask here is: "Can we do similar simplifications and devise a successful algorithm if F is defined in a piecewise fashion, or is otherwise non-smooth?" We consider several examples.

See `http://interval.louisiana.edu/preprints/nonsmooth_degree.ps` or `http://interval.louisiana.edu/preprints/nonsmooth_degree.pdf` for a preprint that contains additional details.

References

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