The Early Days of Interval Global Optimization

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The use of interval arithmetic for global optimization over an n-dimensional interval X was first described by Ramon Moore in a technical report from Stanford University in 1962. The knowledge of the technique became widespread in 1966 through the classical book of Moore, "Interval Analysis" [1].

Section 6.4, "Determination and use of extreme values of rational functions", has all the essentials needed for global optimization: reduction by monotonicity if possible, stationary point reduction by interval Newton if possible, and the use of a splitting strategy if none of the above is applicable.

The splitting strategy is described in Chapter 4 in [1]. It is expressed as a general formula for the computation of an arbitrarily good approximation of the united extension,

$$\bar{f}(X_1, X_2, \ldots, X_n) = \{f(x_1, x_2, \ldots, x_n) | x_i \in X_i,\ i = 1, 2, \ldots, n\}$$

Define

$$X_{i,j}^{(N)} = a_i + [j-1, j](b_i - a_i)/N,\ j = 1, 2, \ldots, N \quad \text{where } X_i = [a_i, b_i],\ a_i \le b_i$$

Using the natural interval extension of f denoted $F(X_1, X_2, \ldots, X_n)$, Theorem 4.4 in [1] states,

$$\begin{aligned} F^{(N)}(X_1, X_2, \ldots, X_n) &= \\ \bigcup_{j_1=1}^{N} \bigcup_{j_2=1}^{N} \cdots \bigcup_{j_n=1}^{N} F(X_{1,j_1}^{(N)}, X_{2,j_2}^{(N)}, \ldots, X_{n,j_n}^{(N)}) &= \bar{f}(X_1, X_2, \ldots, X_n) + E_N \quad (1) \end{aligned}$$

where the width of E_N fulfils $w(E_N) \le (K/N) \max_i(w(X_i))$ for some $K > 0$ and $0 \in E_N$.

The computational cost of (1) is N^n interval function evaluations. If the interval E_N is too wide, it may be reduced by a factor of 2 by computing (1) using $2N$ for N. The cost is now $(2N)^n = 2^n N^n$. A straightforward computation of (1) is therefore mainly of theoretical interest.

In his 1974 paper "Computation of rational interval functions" [2], Skelboe suggested an efficient algorithm for the evaluation of (1). Consider a sequence of computations, $F^{(1)}(X_1, X_2, \ldots, X_n)$, $F^{(2)}(X_1, X_2, \ldots, X_n)$, $F^{(4)}(X_1, X_2, \ldots, X_n)$, $\ldots$ In the computing of $F^{(2N)}$ from $F^{(N)}$, the aim is only to compute those values $F(X^{(2N)}_{1,j_1}, X^{(2N)}_{2,j_2}, \ldots, X^{(2N)}_{n,j_n})$ that affect the interval end values of $F^{(2N)}$. It was also noted that $w(E_N) \le (K/N^2)\max_i(w(X_i))$ when (1) is based on the centered form in stead of the natural interval extension.

The algorithm is closely related to a branch-and-bound algorithm and it is described in [2] both informally and as an Algol program. Today it is often referred to as the Moore-Skelboe algorithm.

The straightforward computation of (1) for $F^{(1)}$, $F^{(2)}$, $F^{(4)}$, $\ldots$ can be structured as a tree, and for most functions it is obvious that the branch-and-bound algorithm only uses the same interval computations for the minimum and maximum interval values during the first few levels of computation of the tree. Therefore the algorithm in [2] first searched for the the minimum and then for the maximum. Some interval function computations are performed twice resulting in a minor waste. Originally the main purpose was to save space, but the strategy also leads to a simpler algorithm.

Moore found interest in this efficient algorithm, and we had an exciting visit of him in Copenhagen in 1975 where we discussed these matters. In his 1976 paper "On computing the range of a rational function of n variables over a bounded region" [3], Moore combined Skelboe's strategy with monotonicity tests, the centered form, and Krawzyk's version of Newton's method into a method for global optimization.

A key observation in [3] is that during the computation of the minimum (or maximum), just one interval function value $F(X^{(N)}_{1,j_1}, X^{(N)}_{2,j_2}, \ldots, X^{(N)}_{n,j_n})$ defines the minimum (or maximum). Therefore it is of interest to compute this interval value as accurately as possible, and this can be done using the appropriate method or combination of methods among the above-mentioned.

The total computational cost in interval function evaluations for (1) using the centered form with a resulting accuracy of $w(E_N) \approx \varepsilon$ is given in [3] as: $(K_1/\varepsilon)^{n/2}$. The branch-and-bound algorithm in [2] requires $K_2 2^n \log_2(K_1/\varepsilon)$ interval function evaluations assuming only a finite number of isolated extrema.

The phrase "Global Optimization" was not used in the early papers. The first time we have seen it is in Eldon Hansen's paper "Global optimization using interval analysis: The one-dimensional case" [4] and in the paper "True worst-case analysis of linear electrical circuits by interval arithmetic" [5] by Skelboe, both published in 1979.

In [5] the algorithm from [2] – augmented with check for monotonicity – is used for the solution of a more realistic problem, namely worst-case analysis of the frequency response of two simple filters with 3 and 4 interval variables, respectively. The interval extension is computed using the mean-value form. This gives the same convergence properties as the centered form, but the mean-value form has two advantages over the centered form: it is straightforward to derive and with the partial derivatives readily available, monotone intervals are identified for free.

In the talk we describe these early attempts, seen from our side. Highlights were visits of Ramon Moore as well as Eldon Hansen, and some years later – in 1980 – a full year's visit of Louis Rall. They all gave us new insight and inspiration through our many discussions and their excellent lectures.

References

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