# Twin Estimates for Slopes

Humberto Muñoz and R. B. Kearfott
Department of Mathematics
University of Louisiana at Lafayette
Box 4-1010, Lafayette, LA 70504-1010 USA
hnm2294@louisiana.edu
rbk@louisiana.edu

## 1 Introduction

Interval slopes are useful in the rigorous treatment of non-smooth optimization problems, as we have outlined in [3, Ch. 6]. There, we developed formulas for outer estimates for the ranges of slopes of non-smooth and discontinuous functions. Slopes of non-smooth functions provide a simple alternative, computable with automatic differentiation procedures, to concepts such as the generalized gradient [1], and semigradient [5].

For various reasons, it may also be useful to compute inner estimates to the range of slopes for non-smooth or for discontinuous functions. For example, we can develop theory relating slopes to generalized gradients; depending on that theory, inner slope estimates would then be guaranteed to be elements of the generalized gradient. We could then develop general, automatic algorithms based on previous algorithms that utilized generalized gradients.

An alternate reason for developing inner estimates is to obtain bounds on the overestimation in the outer slopes.

Formulas for inner estimates for slopes are somewhat trickier than formulas for outer estimates. In [6], such formulas for inner slopes for various elementary functions, such as max and $|\cdot|$ are presented. The development there is analogous to that of [3, Ch. 6].

Incorporation of the formulas for inner slopes into expressions for objective functions, etc. requires an arithmetic based on inner estimations, rather than standard interval arithmetic. We have used *twin arithmetic* as Kreinovich and Nesterov [4, 7] have proposed. This arithmetic is operationally equivalent to Kaucher arithmetic (ibid.).

## 2 A Few Details

We term our procedure automatic twin slope computation (ATSC). Inner and outer bounds of the actual slope set are given simultaneously for nonsmooth functions such as $|f(x)|$, $\max\{f(x), g(x)\}$, $f, g : \mathbb{R}^n \to \mathbb{R}$, and expressions defined by `if-then-else` branches.

**Definition 1** *(Twin arithmetic [4] and [7]). A* twin *is a pair of intervals* $\boldsymbol{t} = (\boldsymbol{x}_{inn}, \boldsymbol{x})$, *with associated relations* $\subseteq$ *and* $\sqsubseteq$, *such that* $\boldsymbol{x}_{inn} \in \mathbb{IR} \cup \{\emptyset\}$, $\boldsymbol{x} \in \mathbb{IR}$, *and for* $\boldsymbol{y} \in \mathbb{IR}$, $\boldsymbol{y} \sqsubseteq \boldsymbol{t}$ *denotes* $\boldsymbol{x}_{inn} \subseteq \boldsymbol{y} \subseteq \boldsymbol{x}$. $(\boldsymbol{x}, \boldsymbol{x})$ *is a degenerate twin, and* $\boldsymbol{y} \sqsubseteq (\emptyset, \boldsymbol{x})$ *means that there is only an outer estimation of* $\boldsymbol{y}$, *which is* $\boldsymbol{x}$.

Basically, a twin estimation of some function $f(x_1, \ldots, x_n)$ consists of a pair of intervals, the inner interval estimation, $f_{inn}(\boldsymbol{t})$ and the outer interval estimation $f(\boldsymbol{x})$. An inner interval estimation must only contain values that are in the actual range of $f$. We denote the twin estimation of $f$ by

$$f_{twin}(\boldsymbol{t}) = (f_{inn}(\boldsymbol{t}), f(\boldsymbol{x})).$$

The basic arithmetic operations with twins given in [7] are identical with those given in Kaucher arithmetic [2] for the set of proper intervals, i.e., $[a, b]$, where $a < b$.

ATSC evaluates functions specified by algorithms or formulas in such a way that all operations are executed according to the rules of a *twin slope arithmetic* to guarantee inner and outer estimations for the function and slope values. Throughout, $\check{\boldsymbol{x}} = (\check{\boldsymbol{x}}_1, \check{\boldsymbol{x}}_2)$ and $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2)$ will represent twins such that $\check{\boldsymbol{x}}_2 \subseteq \boldsymbol{x}_2$.

**Definition 2** *Let* $\boldsymbol{x}$ *and* $\check{\boldsymbol{x}}$ *be real twins and let* $u : \boldsymbol{x}_2 \to \mathbb{R}$ *be a real function. A twin slope for* $u$ *over* $\boldsymbol{x}$ *and centered at* $\check{\boldsymbol{x}}$ *is defined as the twin*

$$\mathbf{S_{twin}}(u, \boldsymbol{x}, \check{\boldsymbol{x}}) = (\mathbf{S_{inn}}(u, \boldsymbol{x}, \check{\boldsymbol{x}}), \mathbf{S}(u, \boldsymbol{x}_2, \check{\boldsymbol{x}}_2)),$$

*where* $\mathbf{S_{inn}}(u, \boldsymbol{x}, \check{\boldsymbol{x}})$ *and* $\mathbf{S}(u, \boldsymbol{x}_2, \check{\boldsymbol{x}}_2)$, *the inner and outer slope estimations, are obtained according to the rules of a twin slope arithmetic.*

Twin slope arithmetic is based on defining operations and standard functions on automatic twin ordered triplets of the form $\langle\langle \check{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{u}^{(s)} \rangle\rangle$, where $\check{\boldsymbol{u}}$, $\boldsymbol{u}$, and $\boldsymbol{u}^{(s)}$ are real twins. $\check{\boldsymbol{u}}$ is the twin evaluation of $u(x)$ over $\check{\boldsymbol{x}}$, $\boldsymbol{u}$ is the twin evaluation of $u(x)$ over $\boldsymbol{x}$ and $\boldsymbol{u}^{(s)}$ is the twin slope $\mathbf{S_{twin}}(u, \boldsymbol{x}, \check{\boldsymbol{x}})$. Inner estimates for slopes are expressed in terms of bounds of intervals, considering concavity conditions of the functions, and executing all intermediate operations with inward rounding. Outer estimates for slopes are obtained with the formulas given in [3] with outward rounding. The following example illustrates the application of ATSC.

**Example 1** *Let* $f(x) = x^2 - 4x + 2$. *Considering the interval* $[1, 7]$ *and its midpoint 4, the actual slope is* $\mathbf{S}^\sharp(f, [1, 7], 4) = [1, 7]$, *and the actual range is* $f^u([1, 7]) = [-2, 23]$. *Let* $\boldsymbol{x} = ([1, 7], [1, 7])$ *and* $\check{\boldsymbol{x}} = ([4 - \epsilon, 4 + \epsilon], [4 - \epsilon, 4 + \epsilon])$, *where* $\epsilon$ *is large enough so repeated inward rounding does not result in the empty set. The next table presents intermediate evaluations using twin arithmetic and twin slope arithmetic with forward substitution. In this table,* $\boldsymbol{xr}$, $\check{\boldsymbol{x}}$, *and* $\boldsymbol{xs}$ *denote the range, center and twin slope evaluations for the intermediate variables respectively (rounded out or in as appropriate to three digits). Also,* $\boldsymbol{op}$ *indicates which intermediate operation is performed to compute the displayed result.*

| ***op*** |
|---|
| $x_1 = x$ |
| $x_2 = x_1^2$ |
| $x_3 = 4x_1$ |
| $x_4 = x_2 - x_3$ |
| $x_5 = x_4 + 2$ |

| $\boldsymbol{op}$ | $\boldsymbol{xr}$ | $\check{\boldsymbol{x}}$ | $\boldsymbol{xs}$ |
|---|---|---|---|
| $x_1$ | $([1,7],[1,7])$ | $([4,4],[4,4])$ | $([1,1],[1,1])$ |
| $x_2$ | $([1,49],[1,49])$ | $([16,16],[15.9,16.1])$ | $([5,11],[4.99,11.1])$ |
| $x_3$ | $([4,28],[4,28])$ | $([16,16],[15.9,16.1])$ | $([4,4],[3.99,4.01])$ |
| $x_4$ | $([-3,21],[-27,45])$ | $([0,0],[-0.01,0.01])$ | $([1.01,6.99],[.999,7.01])$ |
| $x_5$ | $([-1,23],[-25,47])$ | $([2,2],[1.99,2.01])$ | $([1.01,6.99],[.999,7.01])$ |

*Finally the twin slope, and the twin enclosures evaluation with 15 digits in the computation, are*

$$\mathbf{S_{twin}}(f,\boldsymbol{x},\check{\boldsymbol{x}}) = \begin{array}{l} ([1.00000000000001, 6.99999999999997], \\ \;[0.999999999999982, 7.00000000000004])\,, \end{array}$$

$$f_{twin}(\boldsymbol{x}) = \begin{array}{l} ([-.99999999999997, 22.9999999999997], \\ \;[-25.0000000000001, 47.00000000000004])\,. \end{array}$$

# References


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