# Methods of Proving Chaos in Dynamical Systems via Transfer Maps

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The determination of regularity versus chaoticity of motion around fixed points is one of the important questions in the theory of dynamical systems. We define chaoticity as motion in which closeby points in phase space exhibit an exponential growth of separation over a sufficiently large time. The goal of this paper is to determine whether the region around a fixed point of the motion exhibits chaoticity or not, and to study at what values of parameters of the system a transition between the two phenomena develops and the system shows bifurcation. We develop methods that allow a rigorous determination of domains of chaoticity around a fixed point of the motion. Also, in cases where the fixed point is a stable attractor, we will be able to prove this stability. To this end, we begin by determining the flow of the motion and its dependence on parameters as a Taylor model using the verified integrator VI, a so-called transfer map. This Taylor model describes the dependence of final coordinates on initial coordinates and parameters via a Taylor polynomial, and provides a rigorous error of this approximation. We then use methods of verified inversion to determine the fixed point of the motion and its dependence on the parameter. Next, a coordinate shift to this parameter dependent fixed point is performed, and as a result, we have a Taylor model describing the dynamics as a function of parameters that is origin preserving.

In addition to the Taylor model of the flow, we also determine the Taylor model of the Jacobian of the flow as a function of both position and parameter by integrating the respective differential equations with the verified integrator. Then the Eigenvalues of the Jacobian are determined using Taylor model arithmetic, where the ability to suppress the dependency problem of Taylor models proves useful in the linear algebra involved in this step. Finally, the moduli of the Eigenvalues are bounded, and regions are determined in which they are bounded below and above by 1, corresponding to the sought regions of chaoticity and stability, respectively.

The method is illustrated with various examples, including the Henon map and the study of dynamics around Lagrangian points in the Earth-Sun system.