The Lorenz Attractor Exists – An Auto-Validated Proof

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Abstract

We present an algorithm for computing rigorous solutions to a large class of ordinary differential equations. The main algorithm is based on a partitioning process and the use of interval arithmetic with directed rounding. As an application, we prove that the Lorenz equations support a strange attractor, as conjectured by Edward Lorenz in 1963. This conjecture was recently listed by Steven Smale as one of several challenging problems for the 21st century. We also prove that the attractor is robust, i.e., it persists under small perturbations of the coefficients in the underlying differential equations. Furthermore, the flow of the equations admits a unique SRB measure, whose support coincides with the attractor. The proof is based on a combination of normal form theory and rigorous computations.

1 Background to the Problem

The following non-linear system of differential equations,

$$\begin{aligned} \dot{x}_1 &= -\sigma x_1 + \sigma x_2 \\ \dot{x}_2 &= \varrho x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 &= -\beta x_3 + x_1 x_2, \end{aligned} \tag{1}$$

was introduced in 1963 by Edward Lorenz, see [5]. As a crude model of atmospheric dynamics, these equations led Lorenz to the discovery of sensitive dependence of initial conditions - an essential factor of unpredictability in many systems. Numerical simulations for an open neighbourhood of the classical parameter values $\sigma = 10, \beta = 8/3$ and $\varrho = 28$ suggest that almost all points in phase space tend to a strange attractor - *the Lorenz attractor*.

For $\varrho > 1$, there are three fixed points: the origin and the two "twin points"

$$C^{\pm} = (\pm\sqrt{\beta(\varrho - 1)}, \pm\sqrt{\beta(\varrho - 1)}, \varrho - 1).$$

Numerical experiments indicate the existence a forward invariant open set U containing the origin but bounded away from the fixed points $C^{\pm}$. If we let φ denote the flow of (1), we can form the maximal invariant set

$$\mathcal{A} = \bigcap_{t \geq 0} \varphi(U, t).$$

Due to the flow being dissipative, the attracting set $\mathcal{A}$ must have zero volume. It must also contain the unstable manifold of the origin $W^u(0)$, which seems to spiral around $C^{\pm}$ in a very complicated, non-periodic fashion, see Figure 1(a). In particular, $\mathcal{A}$ contains the origin itself, and therefore the flow on $\mathcal{A}$ can not have a hyperbolic structure. The reason is that fixed points of the vector field generate discontinuities for the return maps, and as a consequence, the hyperbolic splitting is not continuous. Apart from this, the attracting set appears to have a strong hyperbolic structure as described below.

As it was very difficult to extract rigorous information about the attracting set $\mathcal{A}$ from the differential equations themselves, a *geometric model* of the Lorenz flow was introduced by John Guckenheimer in the late sixties, see [2]. This model has been extensively studied, and it is well understood today, see e.g. [3], [14], [12], [8], [9], [10]. Oddly enough, the original equations introduced by Lorenz have remained a puzzle. A few computer-assisted proofs, however, have quite recently been announced, see [1], [4], and [6]. These articles deal with subsets of $\mathcal{A}$ which are not attracting, and therefore only concern a set of trajectories having measure zero. Despite this, it has always been widely believed that the flow of the Lorenz equations has the same qualitative behaviour as its geometric model. We prove that the geometric model does indeed give an accurate description of the dynamics of (1).

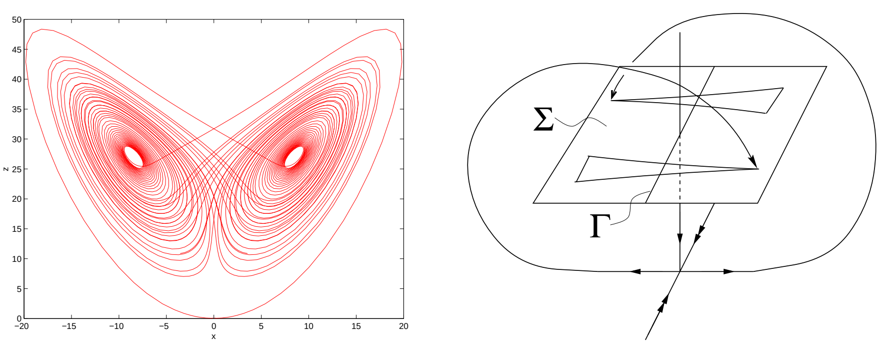

Figure 1: (a) A part of the unstable manifold of the origin. (b) The return map acting on Σ.

By the use of a Poincaré section, the flow of (1) can be reduced to a *first return map* R acting on the section $\Sigma \subset \{x_3 = \varrho - 1\}$, as schematically illustrated in Figure 1(b).

Note that R is not defined on the line $\Gamma = \Sigma \cap W^s(0)$: these points tend to the origin, and never return to Σ. Due to the fixed point at the origin, the return times are not bounded. This constitutes a serious obstruction to any numerical approach. This is overcome by introducing a local change of coordinates, and we prove the following properties of the return map R:

- There exists a compact set $N \subset \Sigma$ such that $N \setminus \Gamma$ is *forward invariant* under R, i.e., $R(N \setminus \Gamma) \subset \mathrm{int}(N)$. This ensures that the flow has an attracting set $\mathcal{A}$ with a large basin of attraction. We can then form a cross-section of the attracting set: $\mathcal{A} \cap \Sigma = \cap_{n=0}^{\infty} R^n(N) = \Lambda$.

- On N, there exists a cone field $\mathfrak{C}$ which is mapped strictly into itself by DR, i.e., for all $x \in N$, $DR(x) \cdot \mathfrak{C}(x) \subset \mathfrak{C}(R(x))$. The cones of $\mathfrak{C}$ are centered along two curves which approximate Λ, and each cone has an opening of at least $5°$.

- The tangent vectors in $\mathfrak{C}$ are eventually expanded under the action of DR: there exists $C > 0$ and $\lambda > 1$ such that for all $v \in \mathfrak{C}(x)$, $x \in N$, we have $|DR^n(x)v| \geq C\lambda^n|v|$, $n \geq 0$. In fact, the expansion is strong enough to ensure that R is topologically transitive on Λ.

The proof can be broken down into two main sections: one global part, which involves finding enclosures to solutions of ODEs, and one local part, which is based on normal form theory. Both parts require the use of interval arithmetic, as described in [7].

2 The Main Result

In a recent issue of *the Mathematical Intelligencer* the Fields medalist Steven Smale presented a list of challenging problems for the 21th century, see [11]. Problem number 14 reads as follows:

> *Is the dynamics of the ordinary differential equations of Lorenz that of the geometric Lorenz attractor of Williams, Guckenheimer, and Yorke?*

By proving the three abovementioned properties of R, we provide an affirmative answer to Smale's question:

Main Theorem *For the classical parameter values, the Lorenz equations support a robust strange attractor $\mathcal{A}$. Furthermore, the flow admits a unique SRB measure μ_φ with* $\mathrm{supp}(\mu_\varphi) = \mathcal{A}$.

In fact, we prove that the attracting set is a singular hyperbolic attractor. Almost all nearby points separate exponentially fast until they end up on opposite sides of the attractor. This means that a tiny blob of initial values rapidly smears out over the entire attractor, as observed in numerical experiments. The complete proof has been published in [13].

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