On the Approximation of Centered Zonotopes in the Plane

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Abstract. The present work is devoted to computation with zonotopes in the plane. Using ideas from the theory of quasivector spaces we formulate an approximation problem for zonotopes and propose an algorithm for its solution.

1 Introduction

Zonotopes are convex bodies with simple algebraic presentation: they are Minkowski sums of segments. By means of a translation (addition of a vector) any zonotope can be centered at the origin, therefore without loss of generality we can restrict our considerations to centered (origin symmetric) zonotopes. The latter are positive combinations of centered unit segments [7], [8]. A fixed system of centered unit segments generates a class of zonotopes consisting of all positive combinations of the unit segments. This class is closed under addition and multiplication by scalar and forms a quasilinear space [2], [5], [6]. A quasilinear space over the field of reals is an additive abelian monoid with cancellation law endowed with multiplication by scalars. Any quasilinear space can be embedded in a group; in addition a natural isomorphic extension of the multiplication by scalars leads to quasilinear spaces with group structure called quasivector spaces [4]. Quasivector spaces obey all axioms of vector spaces, but in the place of the second distributive law we have: $(\alpha + \beta) * c = \alpha * c + \beta * c$, if $\alpha\beta \geq 0$. If, in addition, the elements satisfy the relation: $(-1) * c = c$, then the space is called symmetric quasilinear space.

Every quasivector space is a direct sum of a vector space and a symmetric quasivector space [3], [4]. On the other side, the algebraic operations in vector and symmetric quasivector spaces are mutually representable. This enables us to transfer basic vector space concepts (such as linear combination, basis, dimension, etc.) to symmetric quasivector spaces. Let us also mention that symmetric quasivector spaces with finite basis are isomorphic to a canonic space similar to $\mathbf{R}^n$ [4]. These results can be used for computations with zonotopes as then we practically work in a vector space. Computing with centered zonotopes is especially simple and instructive [1]. In the present work we demonstrated some properties of centered zonotopes in the plane. In particular, an approximation problem related to zonotopes has been formulated and solved by means of a numerical procedure and a MATLAB program. Our procedure allows us to present approximately any centered zonotope in the plane by means of a class of zonotopes over a given basis of centered segments.

2 Quasivector Spaces

By $\mathbf{R}$ we denote the set of reals; we use the same notation for the linearly ordered field of reals $\mathbf{R} = (\mathbf{R}, +, \cdot, \leq)$. For any integer $n \geq 1$ denote by $\mathbf{R}^n$ the set of all n-tuples $(\alpha_1, \alpha_2, ..., \alpha_n)$, where $\alpha_i \in \mathbf{R}$. The set $\mathbf{R}^n$ forms a vector space $\mathbf{V}^n = (\mathbf{R}^n, +, \mathbf{R}, \cdot)$ under addition and multiplication by scalars.

Every abelian monoid $(\mathcal{M}, +)$ with cancellation law induces an abelian group $(\mathcal{D}(\mathcal{M}), +)$, where $\mathcal{D}(\mathcal{M}) = \mathcal{M}^2/\!\sim$ consists of all pairs (A, B) factorized by the congruence relation $\sim$: $(A, B) \sim (C, D)$ iff $A + D = B + C$, for all $A, B, C, D \in \mathcal{M}$. Addition in $\mathcal{D}(\mathcal{M})$ is $(A, B) + (C, D) = (A + C, B + D)$. The null element of $\mathcal{D}(\mathcal{M})$ is the class (Z, Z), $Z \in \mathcal{M}$; we have $(Z, Z) \sim (0, 0)$. The opposite element to $(A, B) \in \mathcal{D}(\mathcal{M})$ is $\mathrm{opp}(A, B) = (B, A)$. All elements of $\mathcal{D}(\mathcal{M})$ admitting the form $(A, 0)$ are called *proper* and the remaining are *improper*. The opposite of a proper element is improper; $\mathrm{opp}(A, 0) = (0, A)$.

Definition 2.1. *Let* $(\mathcal{M}, +)$ *be an abelian monoid with cancellation law. Assume that a mapping "$*$" (multiplication by scalars) is defined on* $\mathbf{R} \times \mathcal{M}$ *satisfying: i)* $\gamma * (A + B) = \gamma * A + \gamma * B$, *ii)* $\alpha * (\beta * C) = (\alpha\beta) * C$, *iii)* $1 * A = A$, *iv)* $(\alpha + \beta) * C = \alpha * C + \beta * C$, *if* $\alpha\beta \geq 0$. *The algebraic system* $(\mathcal{M}, +, \mathbf{R}, *)$ *is called a* quasilinear space over $\mathbf{R}$.

Every quasilinear space $(\mathcal{M}, +, \mathbf{R}, *)$ can be embedded into a group $(\mathcal{D}(\mathcal{M}), +)$. Multiplication by scalars "$*$" is naturally extended from $\mathbf{R} \times \mathcal{M}$ to $\mathbf{R} \times \mathcal{D}(\mathcal{M})$ by means of:

$$\gamma * (A, B) = (\gamma * A, \gamma * B), \quad A,\ B \in \mathcal{M}, \quad \gamma \in \mathbf{R}. \tag{1}$$

In the sequel we shall call quasilinear spaces of group structure, such as $\mathcal{D}(\mathcal{M})$, quasivector spaces, and denote their elements by lower case roman letters, e. g. $a = (A_1, A_2), A_1, A_2 \in \mathcal{M}$.

Definition 2.2. *[4] A* quasivector space (over $\mathbf{R}$)*, denoted* $(\mathcal{Q}, +, \mathbf{R}, *)$*, is an abelian group* $(\mathcal{Q}, +)$ *with a mapping (multiplication by scalars) "$*$":* $\mathbf{R} \times \mathcal{Q} \longrightarrow \mathcal{Q}$*, such that for* $a, b, c \in \mathcal{Q}$, $\alpha, \beta, \gamma \in \mathbf{R}$: $\gamma * (a + b) = \gamma * a + \gamma * b$, $\alpha * (\beta * c) = (\alpha\beta) * c$, $1 * a = a$, $(\alpha + \beta) * c = \alpha * c + \beta * c$, *if* $\alpha\beta \geq 0$.

Proposition 2.1. *[3] Let $(\mathcal{M}, +, \mathbf{R}, *)$ be a quasilinear space over $\mathbf{R}$, and $(\mathcal{Q}, +)$, $\mathcal{Q} = \mathcal{D}(\mathcal{M})$, be the induced abelian group. Let $* : \mathbf{R} \times \mathcal{Q} \longrightarrow \mathcal{Q}$ be multiplication by scalars defined by (1). Then $(\mathcal{Q}, +, \mathbf{R}, *)$ is a quasivector space over $\mathbf{R}$.*

Let a be an element of a quasivector space $(\mathcal{Q}, +, \mathbf{R}, *)$, $a \in \mathcal{Q}$. The operator $\neg a = (-1) * a$ is called *negation*; in the literature it is usually denoted $-a = (-1) * a$. We write $a \neg b = a + (\neg b)$; note that $a \neg a = 0$ may not generally hold. From $\mathrm{opp}(a) + a = 0$ we obtain $\neg \mathrm{opp}(a) \neg a = 0$, that is $\neg \mathrm{opp}(a) = \mathrm{opp}(\neg a)$. We shall use the notation $a_- = \neg \mathrm{opp}(a) = \mathrm{opp}(\neg a)$; the latter operator is called *dualization* or *conjugation*. The relations $\neg \mathrm{opp}(a) = \mathrm{opp}(\neg a) = a_-$ imply $\mathrm{opp}(a) = \neg(a_-) = (\neg a)_-$, shortly $\mathrm{opp}(a) = \neg a_-$. Thus, the symbolic notation $\neg a_-$ can be used instead of $\mathrm{opp}(a)$, and, for $a \in \mathcal{Q}$ we can write $a \neg a_- = 0$, resp. $\neg a_- + a = 0$. We also note that some vector space concepts, such as subspace, sum and direct sum "$\bigoplus$", are trivially extended to quasivector spaces [4]. Some rules for calculation in quasivector spaces are summarized in [4].

Example 1. For any integer $k \geq 1$ the set $\mathbf{R}^k$ of all k-tuples $(\alpha_1, \alpha_2, ..., \alpha_k)$, $\alpha_i \in \mathbf{R}$, with $(\alpha_1, \alpha_2, ..., \alpha_k) = (\beta_1, \beta_2, ..., \beta_k)$ whenever $\alpha_1 = \beta_1, \alpha_2 = \beta_2, ..., \alpha_k = \beta_k$, forms a quasivector space over $\mathbf{R}$ under the operations

$$(\alpha_1, \alpha_2, ..., \alpha_k) + (\beta_1, \beta_2, ..., \beta_k) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, ..., \alpha_k + \beta_k), \tag{2}$$

$$\gamma * (\alpha_1, \alpha_2, ..., \alpha_k) = (|\gamma|\alpha_1, |\gamma|\alpha_2, ..., |\gamma|\alpha_k), \ \gamma \in \mathbf{R}. \tag{3}$$

This quasivector space is denoted by $\mathbf{S}^k = (\mathbf{R}^k, +, \mathbf{R}, *)$. Negation in $\mathbf{S}^k$ is the same as identity while the opposite operator is the same as conjugation:

$$\mathrm{opp}(\alpha_1, \alpha_2, ..., \alpha_k) = (\alpha_1, \alpha_2, ..., \alpha_k)_- = (-\alpha_1, -\alpha_2, ..., -\alpha_k). \tag{4}$$

The direct sum $\mathbf{V}^l \bigoplus \mathbf{S}^k$ of the l-dimensional vector space $\mathbf{V}^l = (\mathbf{R}^l, +, \mathbf{R}, \cdot)$ and the quasivector space $\mathbf{S}^k = (\mathbf{R}^k, +, \mathbf{R}, *)$ is a quasivector space.

Example 2. The system $(\mathcal{K}, +)$ of all convex bodies [7] in a real m-dimensional Euclidean vector space $\mathbf{E}^m$ with addition: $A + B = \{a + b \mid a \in A, b \in B\}$, $A, B \in \mathcal{K}$, is an abelian monoid with cancellation law having as a neutral element the origin "0" of $\mathbf{E}^m$. The system $(\mathcal{K}, +, \mathbf{R}, *)$, where "$*$" is multiplication by real scalars defined by: $\gamma * A = \{\gamma a \mid a \in A\}$, is a quasilinear space (of monoid structure), that is the following four relations are satisfied: i) $\gamma * (A + B) = \gamma * A + \gamma * B$, ii) $\alpha * (\beta * C) = (\alpha\beta) * C$, iii) $1 * A = A$, iv) $(\alpha + \beta) * C = \alpha * C + \beta * C, \quad \text{if} \ \ \alpha\beta \geq 0$ [4]. The monoid $(\mathcal{K}, +)$ induces a group of generalized convex bodies $(\mathcal{D}(\mathcal{K}), +)$, According to Proposition 2.1. the space $(\mathcal{D}(\mathcal{K}), +, \mathbf{R}, *)$, where "$*$" is defined by (1) is a quasivector space [3]. A centrally symmetric convex body with center at the origin will be called *centered* convex body, cf. [7], p. 383. Centered convex bodies do not change under multiplication by -1.

Definition 2.3. *Q is a quasivector space. An element $a \in \mathcal{Q}$ with $a \neg a = 0$ is called* linear. *An element $a \in \mathcal{Q}$ with $\neg a = a$ is called* origin symmetric.

Proposition 2.2. *Assume that Q is a quasivector space. The subsets of linear and symmetric elements $\mathcal{Q}' = \{a \in \mathcal{Q} \mid a \neg a = 0\}$, resp. $\mathcal{Q}'' = \{a \in \mathcal{Q} \mid a = \neg a\}$ form subspaces of $\mathcal{Q}$. The subspace $\mathcal{Q}'$ is a vector space called the vector (linear) subspace of $\mathcal{Q}$.*

The space $\mathcal{Q}'' = \{a \in \mathcal{Q} \mid a = \neg a\}$ of centered elements is called the *symmetric centered subspace* of $\mathcal{Q}$. For symmetric elements $b \in \mathcal{Q}''$ the following relations are equivalent: $b = \neg b \Longleftrightarrow b + b_- = 0 \Longleftrightarrow b_- = \mathrm{opp}(b)$. The following theorem shows the important roles of symmetric quasivector spaces.

Theorem 2.1. *[4] For every quasivector space $\mathcal{Q}$ we have $\mathcal{Q} = \mathcal{Q}' \bigoplus \mathcal{Q}''$. More specifically, for every $x \in \mathcal{Q}$ we have $x = x' + x'' = (x'; x'')$ with unique $x' = (1/2) * (x + x_-) \in \mathcal{Q}'$, and $x'' = (1/2) * (x \neg x) \in \mathcal{Q}''$.*

Symmetric quasivector spaces. Let $(\mathcal{Q}, +, \mathbf{R}, *)$ be a symmetric quasivector space over $\mathbf{R}$. Define the operation "$\cdot$": $\mathbf{R} \times \mathcal{Q} \longrightarrow \mathcal{Q}$ by

$$\alpha \cdot c = \alpha * c_{\sigma(\alpha)} = \begin{cases} \alpha * c, & \text{if } \alpha \geq 0, \\ \alpha * c_-, & \text{if } \alpha < 0, \end{cases} \tag{5}$$

where $\sigma(\gamma) = \{+, \text{if } \gamma \geq 0; -, \text{if } \gamma < 0\}$ and $c_+ = c$.

Theorem 2.2. *[3], [4] Let $(\mathcal{Q}, +, \mathbf{R}, *)$ be a symmetric quasivector space over $\mathbf{R}$. Then $(\mathcal{Q}, +, \mathbf{R}, \cdot)$, with "$\cdot$" defined by (5), is a vector space over $\mathbf{R}$.*

Note that for a centered, the element $(-1) \cdot a = (-1) * a_- = a_-$ is the opposite to a, that is $a + (-1) \cdot a = 0$, resp. $a + a_- = 0$.

Linear combinations. Assume that $(\mathcal{S}, +, \mathbf{R}, *)$ is a *symmetric* quasivector space and $(\mathcal{S}, +, \mathbf{R}, \cdot)$ is the associated vector space from Theorem 2.2. We may transfer vector space concepts from $(\mathcal{S}, +, \mathbf{R}, \cdot)$, such as linear combination, linear dependence, basis etc., to the original symmetric quasivector space $(\mathcal{S}, +, \mathbf{R}, *)$. For example, let $c^{(1)}, c^{(2)}, ..., c^{(k)}$ be finitely many elements of $\mathcal{S}$. The familiar linear combination $f = \sum_{i=1}^{k} \alpha_i \cdot c^{(i)} = \alpha_1 \cdot c^{(1)} + \alpha_2 \cdot c^{(2)} + ... + \alpha_k \cdot c^{(k)}$, $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbf{R}$, in the induced vector space $(\mathcal{S}, +, \mathbf{R}, \cdot)$, can be rewritten using (5) as

$$f = \alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * c^{(2)}_{\sigma(\alpha_2)} + ... + \alpha_k * c^{(k)}_{\sigma(\alpha_k)}. \tag{6}$$

Thus (6) is a *linear combination* of $c^{(1)}, c^{(2)}, ..., c^{(k)} \in \mathcal{S}$ in the symmetric quasivector space $(\mathcal{S}, +, \mathbf{R}, *)$. Similarly, the concepts of spanned subspace, linear (in)dependency, linear mapping, basis, dimension, etc. are defined, and the theory of vector spaces can be reformulated in $(\mathcal{S}, +, \mathbf{R}, *)$ [3], [4].

Theorem 2.3. *[4] Any symmetric quasivector space over $\mathbf{R}$, with a basis of k elements, is isomorphic to $\mathbf{S}^k = (\mathbf{R}^k, +, \mathbf{R}, *)$.*

3 Computation with Centered Zonotopes in the Plane

Using the theory of symmetric quasivector spaces that we briefly outlined in the previous section, we consider next a class of special convex bodies, namely centered zonotopes in the plane.

Every unit vector in the plane $\mathbf{R}^2$: $e = e(\varphi) = (\cos\varphi, \sin\varphi) \in \mathbf{R}^2,\ \varphi \in [0, \pi)$, defines a centered segment $\tilde{e}$ with endpoints $-e$ and e:

$$\tilde{e} = \mathrm{conv}\{-e, e\} = \{\lambda e \mid \lambda \in [-1, 1]\}.$$

For $\rho \in \mathbf{R}$ denote $s = \rho e$. Multiplication of a unit centered segment $\tilde{e}$ by a scalar $\rho \in \mathbf{R}$ is:

$$\tilde{s} = \rho * \tilde{e} = (\rho e)\tilde{} = \mathrm{conv}\{-s, s\} = \{\lambda\rho e \mid \lambda \in [-1, 1]\}.$$

More generally, multiplication of a centered segment $(\rho e)\tilde{}$ by a scalar $\gamma \in \mathbf{R}$ gives $\gamma * (\rho e)\tilde{} = ((\gamma\rho)e)\tilde{}$. Note that $-1 * \tilde{s} = \tilde{s}$; more generally, $(-\rho) * \tilde{s} = \rho * \tilde{s}$ (whereas, for comparison, we have, of course, $(-\rho)s \neq \rho s$).

Minkowski addition of colinear centered segments is $(\rho_1 e)\tilde{} + (\rho_2 e)\tilde{} = ((\rho_1 + \rho_2)e)\tilde{}$. To present Minkowski addition of noncolinear centered segments, assume $0 \leq \varphi_1 < \varphi_2 < \pi$ and denote $e^{(1)} = (\cos\varphi_1, \sin\varphi_1), e^{(2)} = (\cos\varphi_2, \sin\varphi_2)$. The points

$$s^{(1)} = \rho_1 e^{(1)} = (\rho_1 \cos\varphi_1, \rho_1 \sin\varphi_1),$$
$$s^{(2)} = \rho_2 e^{(2)} = (\rho_2 \cos\varphi_2, \rho_2 \sin\varphi_2),$$

$\rho_1, \rho_2 \in \mathbf{R}$, define two noncolinear centered segments $\tilde{s}^{(1)}, \tilde{s}^{(2)} \in \mathbf{R}^2$. The Minkowski sum $\tilde{s}^{(1)} + \tilde{s}^{(2)}$ is a centered quadrangle (parallelepiped) P with vertices $\{t^{(1)}, t^{(2)}, -t^{(1)}, -t^{(2)}\}$, where $t^{(1)} = \ \ s^{(1)} + s^{(2)},\ t^{(2)} = -s^{(1)} + s^{(2)}$. The perimeter of $P = \mathrm{conv}\{t^{(1)}, t^{(2)}, -t^{(1)}, -t^{(2)}\}$ is $4(\rho_1 + \rho_2)$ and the area of P is $4\rho_1\rho_2 \sin(\varphi_2 - \varphi_1)$.

Assume that we are given k fixed centered unit vectors $e^{(1)}, e^{(2)}, ..., e^{(k)} \in \mathbf{R}^2$ in cyclic anticlockwise order, such that $0 \leq \varphi_1 < \varphi_2 < ... < \varphi_k < \pi$.

A system $\{e^{(i)}\}_{i=1}^k$ of centered unit vectors $e^{(i)} = e(\varphi_i) \in \mathbf{R}^2$ satisfying $0 \leq \varphi_1 < ... < \varphi_k < \pi$ will be further called *regular*; the same notion will be used for the induced system of centered unit segments $\{\tilde{e}^{(i)}\}$. In particular, the system $\{e(\varphi_i)\}$ with $\varphi_i = \pi(i-1)/k,\ \ i = 1, ..., k$, is regular; for this system we have $\varphi_{i+1} - \varphi_i = \pi/k = \mathrm{const}$.

For $\alpha_i \geq 0$ the vectors $s^{(i)} = \alpha_i e^{(i)} = (\alpha_i \cos\varphi_i, \alpha_i \sin\varphi_i)$ induce the centered segments $\tilde{s}^{(i)} = \alpha_i * \tilde{e}^{(i)} = (\alpha_i e^{(i)})\tilde{},\ \ i = 1, ..., k$. The *positive combination* of the segments $\tilde{s}^{(i)}$

$$\tilde{z} = \sum_{i=1}^{k} \tilde{s}^{(i)} = \sum_{i=1}^{k} \alpha_i * \tilde{e}^{(i)},\ \alpha_i \geq 0, \tag{7}$$

is a centered *zonotope* with $2k$ vertices: $t^{(1)}, t^{(2)}, ..., t^{(k)}, -t^{(1)}, -t^{(2)}, ..., -t^{(k)}$ [7], [8], where

$$\begin{aligned} t^{(1)} &= \ \ \alpha_1 e^{(1)} + \alpha_2 e^{(2)} + ... + \alpha_{k-1} e^{(k-1)} + \alpha_k e^{(k)}, \\ t^{(2)} &= -\alpha_1 e^{(1)} + \alpha_2 e^{(2)} + ... + \alpha_{k-1} e^{(k-1)} + \alpha_k e^{(k)}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ t^{(i)} &= -\alpha_1 e^{(1)} - ... - \alpha_{i-1} e^{(i-1)} + \alpha_i e^{(i)} + ... + \alpha_k e^{(k)}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ t^{(k)} &= -\alpha_1 e^{(1)} - \alpha_2 e^{(2)} + ... - \alpha_{k-1} e^{(k-1)} + \alpha_k e^{(k)}. \end{aligned} \tag{8}$$

The vertices $t^{(1)}, t^{(2)}, ..., t^{(k)}$ given by (8) are lying in cyclic anticlockwise order in a half-plane between the vectors $t^{(1)}$ and $t^{(k)} = -t^{(1)} + 2\alpha_k e^{(k)}$.

Two centered zonotopes $B = \sum_{i=1}^{k} \beta_i * \tilde{e}^{(i)}$, $C = \sum_{i=1}^{l} \gamma_i * \tilde{e}^{(i)}$, $\beta_i \geq 0$, $\gamma_i \geq 0$, over same system $\{\tilde{e}^{(i)}\}_{i=1}^{k}$, are added by

$$B + C = \sum_{i=1}^{k} \beta_i * \tilde{e}^{(i)} + \sum_{i=1}^{l} \gamma_i * \tilde{e}^{(i)} = \sum_{i=1}^{k} (\beta_i + \gamma_i) * \tilde{e}^{(i)}.$$

Thus given a fixed regular system of centered unit segments $\{\tilde{e}^{(i)}\}_{i=1}^{k}$, the set of all zonotopes of the form $A = \sum_{i=1}^{k} \alpha_i * \tilde{e}^{(i)}, \quad \alpha_i \in \mathbf{R}$, is closed under Minkowski addition and multiplication by scalars and forms a quasilinear space (of monoid structure) [3], [4].

Consider two zonotopes $B = \sum_{i=1}^{l} \beta_i * \tilde{u}^{(i)}$, $C = \sum_{i=1}^{m} \gamma_i * \tilde{v}^{(i)}$, where $\{\tilde{u}^{(i)}\}_{i=1}^{l}$, $\{\tilde{v}^{(i)}\}_{i=1}^{m}$, are two distinct regular systems of centered unit segments. From the expression for the sum: $B + C = \sum_{i=1}^{l} \beta_i * \tilde{u}^{(i)} + \sum_{i=1}^{m} \gamma_i * \tilde{v}^{(i)}$ we see that the vertices of $B + C$ can be restored using (8), where $\{\tilde{e}^{(i)}\}_{i=1}^{k}$ is the union of the two sets $\{\tilde{u}^{(i)}\}_{i=1}^{l}$, $\{\tilde{v}^{(i)}\}_{i=1}^{m}$ (after proper ordering). Clearly the number of vertices of $B + C$ equals (generally) the sum $l + m$ of the numbers of vertices of B and C, resp. If we want to use a fixed presentation of the zonotopes of the form (7), then we need to present (approximately) all zonotopes using one and the same system of centered unit segments.

4 An Approximation Problem

The problem. Assume that a regular system of centered unit segments $\tilde{e}^{(1)}, ..., \tilde{e}^{(k)}$, $e^{(i)} = (\cos\varphi_i, \sin\varphi_i)$, $i = 1, ..., k$ is given, which will be considered as basic. Assume that $\{\tilde{p}^{(i)}\}_{i=1}^{m}$ is a regular system of unit centered segments, distinct from the given system $\{\tilde{e}^{(i)}\}_{i=1}^{k}$. We want to approximate a given zonotope of the form $w = \sum_{i=1}^{m} \rho_i * \tilde{p}^{(i)}, \rho_i \geq 0$, by means of zonotopes from the class $z = \sum_{i=1}^{k} \varepsilon_i * \tilde{e}^{(i)}$, so that $w \subseteq z$.

The algorithm. Given are unit vectors $p^{(i)} = (\cos\psi_i, \sin\psi_i)$, $i = 1, ..., m$, such that $0 \leq \psi_i < \psi_{i+1} < \pi$, and nonnegative numbers $\rho_i \geq 0, i = 1, ..., m$, defining a zonotope $w = \sum_{i=1}^{m} \rho_i * \tilde{p}^{(i)}$. We want to find suitable values $\{\varepsilon_i\}_{i=1}^{k}$

such that the zonotope $z = \sum_{i=1}^{k} \varepsilon_i * \tilde{e}^{(i)}$ is an outer approximation of w, that is $w \subseteq z$.

We present the vector $p^{(i)} = (\cos\psi_i, \sin\psi_i)$ as

$$p^{(i)} = \varepsilon_{i1} e^{(j)} + \varepsilon_{i2} e^{(j+1)},\ j = j(i),\ i = 1, ..., m, \tag{9}$$

where $e^{(j)},\ e^{(j+1)}$ are the nearest basic unit vectors enclosing $p^{(i)}$ with $\varphi_j \le \psi_i \le \varphi_{j+1}$ and $\varepsilon_{i1},\ \varepsilon_{i2}$ are some nonnegative coeficients. Clearly, relations (9) define the coefficients $\varepsilon_{i1},\ \varepsilon_{i2}$ in a unique way. We note that if some of the equalities $\varphi_j = \psi_i$, $\psi_i = \varphi_{j+1}$ takes place, then one of the coefficients $\varepsilon_{i1},\ \varepsilon_{i2}$ will be equal to zero.

Relations (9) imply the inclusions $\tilde{p}^{(i)} \subseteq \varepsilon_{i1} * \tilde{e}^{(j)} + \varepsilon_{i2} * \tilde{e}^{(j+1)},\ i = 1, ..., m$. Subsequently we obtain

$$w = \sum_{i=1}^{m} \rho_i * \tilde{p}^{(i)} \subseteq \sum_{i=1}^{m} \rho_i * (\varepsilon_{i1} * \tilde{e}^{(j)} + \varepsilon_{i2} * \tilde{e}^{(j+1)}) = \sum_{i=1}^{k} \varepsilon_i * \tilde{e}^{(i)} = z, \tag{10}$$

with some $\varepsilon_i \ge 0$ that can be effectively computed.

It follows from (10) that the zonotope z is an outer approximation of the zonotope w. From (10) one can compute the vertices of the zonotope z by means of (8). It can be shown that the above algorithm produces an optimal approximation as regard the Hausdorff/integral metric. Roughly specking this follows from the fact that every single segment $\tilde{p}^{(i)}$ has been optimally approximated. Note that the area of the zonotope $\varepsilon_{i1} * \tilde{e}^{(j)} + \varepsilon_{i2} * \tilde{e}^{(j+1)}$ is $4\varepsilon_{i1}\varepsilon_{i2}\sin(\varphi_{j+1} - \varphi_j)$. This can be used to compute the area of the zonotope z.

Relations (9), (10) define an algorithm leading to the construction of an outer approximation z of w. Such an algorithm using a uniform mesh of centered segments in $\mathbf{R}^2$ of the form $s^{(i)} = (\cos\varphi_i, \sin\varphi_i),\ \varphi_i = \pi(i-1)/k,\ i = 1, ..., k$, has been realized in MATLAB. It has been demonstrated that when the number k of mesh points increases, the zonotope z approaches the original zonotope w in Hausdorff sense.

The method of support functions. Let us compare our method with a corresponding method based on support functions. The support function of a set $A \in \mathbf{R}^2$ is $h(A; u) = \max_{x \in A} \langle x, u \rangle$; this function is well defined by its values on the unit circle C. For $u = (\cos\theta, \sin\theta) \in C$ we can easily calculate $h(A; u)$ for any zonotope A. Let first A be a centered unit segment $\tilde{e}$ with $e(\varphi) = (\cos\varphi, \sin\varphi)$. We have $h(\tilde{e}; u) = h(\tilde{e}(\varphi); u) =\mid \cos\varphi\cos\theta + \sin\varphi\sin\theta \mid=\mid \cos(\theta - \varphi) \mid$.

Similarly we can write down the support function of the zonotope $z = \sum_{i=1}^{k} \varepsilon_i * \tilde{e}^{(i)}$, where $e^{(i)} = (\cos\varphi_i, \sin\varphi_i)$, $i = 1, ..., k$, $0 \le \varphi_1 \le ... \le \varphi_m < \pi$. We have $h(z, \theta) = \sum_{i=1}^{k} \varepsilon_i h(\tilde{e}^{(i)}; \theta) = \sum_{i=1}^{k} \varepsilon_i \mid \cos(\theta - \varphi_i) \mid$.

The above approximation problem can be stated in terms of support functions as follows. Given some $\psi_i, i = 1, ..., m$, $0 \le \psi_1 \le ... \le \psi_m < \pi$, we want to approximate from above in the interval $[0, \pi]$ the function $w(\theta) = \sum_{i=1}^{m} \varepsilon_i \mid \cos(\theta - \psi_i) \mid$ by means of a function of the class $z(\theta) = \sum_{i=1}^{k} \varepsilon_i \mid \cos(\theta - \varphi_i) \mid$, that is we need to find the ε's in z so that $z \ge w$ and z approximates w.

Concluding Remarks

From the theory of quasivector spaces it follows that we can compute with zonotopes as we do in vector spaces unless the zonotopes belong to the same space, that is to be presented by the same basic system. This makes it important to be able to approximately present any zonotope by a zonotope from a chosen basic class. We construct a simple algorithm for this purpose which can be effectively implemented in a software environment. Our approach is alternative to the approach based on support functions, extensively used in the literature on convex bodies and seems to be more direct and more simple. From the formulation of the approximation problem in terms of support functions we see that the latter formulation is in no way easier than the direct formulation in terms of zonotopes. We thus conclude that in some cases direct computation with zonotopes may be preferable that related computations based on support functions. Due to their simple presentation zonotopes can be used instead of more traditional convex bodies like boxes, parallelepipeds, ellipsoids etc.

Acknowledgments

This work was supported in part by Bulgarian NSF grants I-903/99, MM-1104/01, and the Swiss NSF Co-operation Agreement No. 7IP 65642 (Program SCOPES).

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