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Risk Analysis Without Monte Carlo Methods

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RISK ANALYSIS WITHOUT MONTE CARLO METHODS

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Dedicated to Prof. Dr. Karl Nickel on the occasion of his 60th birthday.

A new and simple method is presented for computing the cumulative probability distribution of a measure of investment worth (such as rate-of-return). The method involves an arithmetic of inequalities (interval arithmetic) and, unlike Monte Carlo methods, it does not require large samples of points from assumed input distributions. The method can be applied to implicit endogenous variables in economic models as well.

AMS Subject Classification: 65U05, 65G10.

1. Introduction

The nature and uses of risk analysis in project appraisal under uncertainty have been discussed in a number of papers [e.g., Hertz (1979), Pouliquen (1970), and Reutlinger (1970)].

Risk analysis, here as in the cited papers, is a computation of an approximate cumulative probability distribution for a measure of investment worth such as rate-of-return (ROR) based on estimated values, sets of values, ranges of values, or assumed probability distributions for each of the factors upon which ROR depends.

Monte Carlo methods are the standard approach to finding an approximate distribution of ROR. A large number of sample input values are selected from sets of given values or are generated "randomly" according to assumed input distributions. For each of these, ROR is computed. From these results, an approximate distribution of ROR is constructed by a "frequency count" (what fraction of the total number of values of ROR obtained was less than or equal to a given value?). For a large enough sample of input vectors (often in the thousands), the approximation may be adequate.

In this paper, an alternative approach will be presented. It is based on interval arithmetic and interval analysis [e.g., Moore (1979)]. An important feature of the interval

approach is the possibility of greatly reducing the number of computations required to obtain an adequate approximation to the ROR distribution.

There are various measures of investment worth that are used for either of two purposes: (1) to help decide whether or not to make a particular investment, or (2) to help in ranking competing proposed investments. Such decisions arise, in particular, in "capital budgeting" [e.g., Bierman and Smidt (1980), Osteryoung (1979), and Weston and Brigham (1972)].

What we are calling rate-of-return (ROR) here, is also referred to as internal-rate-of-return, yield, interest rate of return, return on investment, present-value return on investment, discounted cash flow, investor's method, time-adjusted rate of return, marginal efficiency of capital, and, perhaps other terminology as well. Financial terminology seems far from uniform, however, and some of the terms above are used in different places to mean different things. For example, "return on investment" is sometimes used to mean ROR and sometimes to mean income divided by investment.

For ranking of investments, at least, it seems that the use of a "discounted cash flow method" is preferred. "Measures of investment worth that do not involve the use of the discounted cash flow method can give rankings of investments that are obviously incorrect" [Bierman and Smidt (1980, p.40)].

Net present value is another discounted cash flow method of measuring investment worth. The "present value profile", which graphs net present value (NPV) versus rate of discount, shows the ROR as the rate of discount at which NPV is zero. Thus, ROR can be viewed as a simplification of the present value profile. The present value profile contains more information than ROR, which is but a single point on the graph.

The techniques presented in Section 2 of this paper could be applied also to obtain distributions for rates of discount corresponding to any net present values simply by adding a constant, NPV, to the equation defining ROR.

Suppose we have estimated costs (investments) C_i and benefits (returns) B_i for a proposed investment in the ith time periods, for $i = 1,2,\ldots,N$. We define the net transfer (cash flow) n_i in the ith time period by $n_i = B_i - C_i$. For a given discount rate, r, the net present value (NPV) is defined by

$$NPV = n_0 + n_1(1+r)^{-1} + \ldots + n_N(1+r)^{-N}, \qquad (1)$$

where n_0 is usually $-C_0$, an initial investment.

ROR is defined as the root of equation (1) corresponding to NPV = 0. In Moore (1979), sufficient conditions are given for the existence of a unique real root of equation (1); [see Chapter 9]. Equation (1) can be written as a polynomial equation by multiplying through by $(1+r)^N$.

In Moore (1979, Chapter 9), an efficient computational method is given for finding the interval of possible values of r satisfying (1) corresponding to assumed intervals of allowed values for each of the n_i. We will reproduce the method as part of a computer program given in the appendix to this paper.

In the final section of this paper, Section 3, we will discuss the use of interval methods in economic models with data that are stochastically related by multivariate probability distributions.

The main results of this paper are presented in Section 2. We introduce a new and simple method for constructing cumulative probability distributions. It provides an approach to risk analysis without the use of costly Monte Carlo methods.

Since techniques are available for computing intervals which contain all the solutions of a linear or nonlinear system of algebraic equations, in the neighborhood of a nominal solution, corresponding to a given set of interval coefficients [e.g., Gay (1983), Moore (1979)], the methods of Section 2 can be applied to finding cumulative probability distributions for implicitly defined endogenous variables in an economic model corresponding to given density functions for the exogenous variables.

2. Interval methods for constructing distributions

An interval of numbers, denoted by [a,b], is a set of numbers between two extreme values, the "low" value a and the "high" value b. Thus, x is in [a,b] if $a \leq x \leq b$. We include the "endpoints" a and b in the interval [a,b].

As in the case of the interval of selling prices of a given stock during a given day or year, there might be only a finite set of different possible values between the low and high values. On the other hand, the set might be a continuum (at least in principle) such as the interval of temperatures at a given place during a given day. Of course, if we measure temperatures to the nearest degree, then this set will be finite also.

Interval arithmetic [e.g., Moore (1979)] is the natural arithmetic for inequalities.

From $a \leq x \leq b$ and $c \leq y \leq d$, we obtain

$$a + c \leq x + y \leq c + d ;$$

therefore, we define the addition of two intervals by

$$[a,b] + [c,d] = [a+c,\ b+d]. \tag{2}$$

If all we know about x is that it is in [a,b] and all we know about y is that is is in [c,d], then all we know about x + y is that it is in [a,b] + [c,d].

Interval subtraction is defined by

$$[a,b] - [c,d] = [a-d,\ b-c], \tag{3}$$

and expresses the fact that if x is in [a,b] and y is in [c,d], then x - y is in the interval defined by (3).

Similarly, for interval multiplication, we define

$$[a,b][c,d] = [\min(ac, ad, bc, bd),\ \max(ac, ad, bc, bd)]. \tag{4}$$

The reciprocal of an interval is defined by

$$1/[c,d] = [1/d,\ 1/c] \quad \text{if 0 is not in } [c,d]. \tag{5}$$

For the division of one interval by another we combine the definitions of the reciprocal and multiplication, thus

$$[a,b]/[c,d] = [a,b](1/[c,d]). \tag{6}$$

The algebraic properties of interval arithmetic, the analysis of interval valued functions, and computational methods for the solution of equations and systems of equations (both linear and nonlinear) with interval coefficients have been discussed in a number of places [e.g., Moore (1979) and the more than 700 references contained therein].

We can allow the endpoints of an interval (the high and low values) to be the same, and we can identify such "degenerate" intervals with ordinary real numbers. Thus we can mix inter-

vals and real numbers in arithmetic computation. For example,

$$[1,2] + 1 = [2,3],$$

and

$$2[3,4] = [6,8].$$

Using interval arithmetic (or interval arithmetic with directed rounding on a computing machine) we can determine, rigorously, upper and lower bounds on the ranges of values of a rational function of one or more variables (lying in intervals) with one evaluation [Moore (1979)].

If each variable occurs only once in an expression for a rational function, then we can obtain the exact range of values in this way. For example, if

$$f = (ay/(bx - c)) - d \tag{7}$$

and we only know that

x is in [1,2]

y is in [12,14]

a is in [.05,.07]

b is in [1.1,1.8]

c is in [.1,.2], and

d is in [1,3];

then, using the rules for interval arithmetic, we find the range of values of f is [-2,829, .089] to three decimal places.

If any variable occurs more than once, then we may overestimate the range of values. Using more sophisticated methods of interval analysis [Moore (1979)], we can keep the amount of overestimation arbitrarily small.

In Moore (1979, Chapter 9), we discussed an evaluation by the World Bank of a proposed forestry project. From a projected cost-benefit stream over a fifty year period, we computed the ROR = .09116 (9.116%). We also computed upper and lower bounds on ROR corresponding to certain variations in the cost-benefit stream. If each of the net transfers is allowed to vary independently up or down by a certain percentage, we can compute, by a single interval computation, the corresponding range of values of ROR. Thus, we obtained the following results with three interval solutions of the ROR equation.

variation in each time period of the net transfer	low ROR	high ROR
± 5%	.085016	.09737
± 10%	.078891	.10366
± 15%	.072760	.11007

We will use the results in the table above for our first illustration of an interval method for constructing distributions in risk analysis.

It has been mentioned [e.g., Pouliquen (1970)] that either discrete or step rectangular distributions (density functions)

of inputs to an economic model are easiest for field appraisers to estimate. Step rectangular distributions are also the most convenient for interval computation of the resulting distributions of ROR.

Suppose we estimate a step rectangular density function to the percentage variations in the cost-benefit stream (net transfer in each time period) for the data presented in the table above. We will consider two cases for illustrating the method of constructing cumulative distributions based on interval computations.

	probability p	range of variations
Case 1)		
	.1	-15% to -10%
	.2	-10% to - 5%
	.4	- 5% to + 5%
	.2	+ 5% to +10%
	.1	+10% to +15%
Case 2)		
	.05	below -15%
	.05	-15% to -10%
	.1	-10% to - 5%
	.6	- 5% to + 5%
	.1	+ 5% to +10%
	.05	+10% to +15%
	.05	above +15%

From the assignment of probabilities for ranges of variations in the cost-benefit stream such as those given in cases 1) and 2) above, we can construct a cumulative probability distribution for ROR based on the intervals between high and low values as shown in the previous table. The results are shown in Figure 1.

[Figure 1, p. 12]

The interval method used to construct the graphs in Figure 1 will now be explained.

In case 1), the given probabilities for the ranges of variations in inputs add to 1.0; therefore we assign probability 0 to ROR values outside the interval [.072760, .11007]. We assign probability 1.0 to ROR being in [.072760, .11007]. Using the computed low and high values for ROR for the various percentage variations in net transfers (inputs), we then construct the following step rectangular density for ROR.

Case 1)	ROR interval	probability
	[.072760, .078891]	.1
	[.078891, .085016]	.2
	[.085016, .09737]	.4
	[.09737 , .10366]	.2
	[.10366 , .11007]	.1

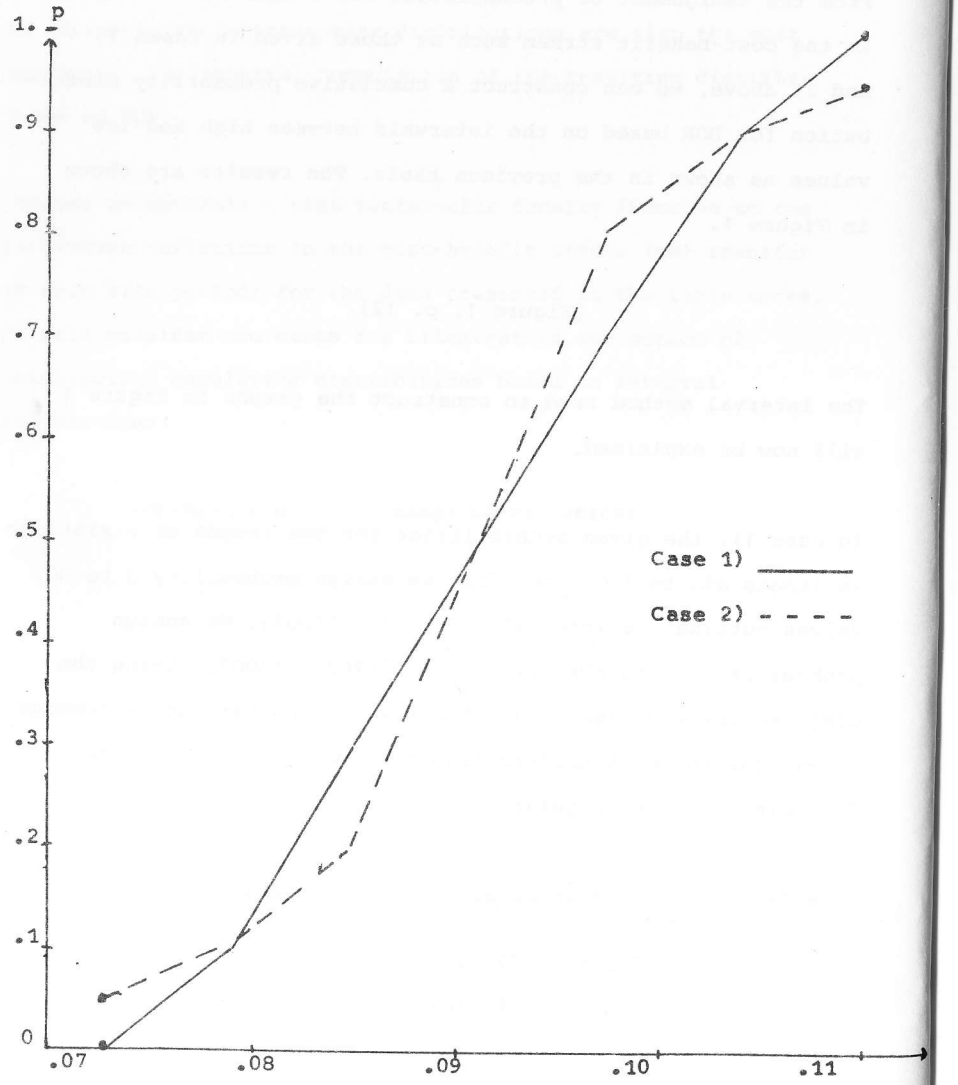

Figure 1 : Cumulative distributions for rate-of-return (ROR) in a proposed highway project

The graph for case 1) in Figure 1 is the integral of the above step rectangular density function. Thus, the vertical axis represents the probability that ROR is less than or equal to a given value on the horizontal axis. Such a graph would be a basis for risk analysis [e.g., Hertz (1979)]. For example, we can read from the graph (in case 1)) that the probability that ROR is at least 8.5% is .7 (= 1.0 - .3).

In case 2) the graph shown in Figure 1 is constructed in a similar way. In this case, we assign probability .05 to the event that ROR is below .072760 and .05 to the event that ROR is above .11007. Otherwise, the construction follows that in case 1), but with the different set of probabilities given for case 2). In case 2), the probability is .8 that ROR is at least 8.5%, according to the graph for case 2) in Figure 1; since the probability that ROR is less than .085 is .2.

Next we present a comparison of the interval approach and the Monte Carlo approach to the construction of a cumulative probability distribution for a very simple example.

Let x and y be independent random variables, each distributed uniformly on the unit interval [0,1]. Then the density function of x + y is triangular; and the cumulative distribution of x + y is expressible as

$$P(x+y \leq r) = \begin{cases} 0, & \text{for } r \leq 0 \\ r^2/2, & \text{for } 0 \leq r \leq 1, \\ 1-(2-r)^2/2, & \text{for } 1 \leq r \leq 2, \\ 1, & \text{for } 2 \leq r. \end{cases} \quad (8)$$

In Figure 2, we show the exact distribution along with four approximating distributions: one using two interval sums, one with four interval sums, one with 50 Monte Carlo sums, and one with 100 Monte Carlo sums. The most accurate [1] of these approximations is the one using four interval sums. Even the one using only two interval sums is closer than some of the Monte Carlo results. We now explain how the graphs in Figure 2 were constructed.

The Monte Carlo sums were generated by rolling dice. The interval of possible values for $x+y$ is $[0,2]$. The 11 possible values for a roll of two dice $(2,3,\ldots,12)$ were distributed uniformly in the corresponding sub-intervals $[0, 2/11]$, $[2/11, 4/11]$, ..., $[20/11, 2]$. Thus, the most likely sum, 7, occurs in $[10/11, 12/11]$. Two separate sequences of rolls were carried out: one with 50 rolls and one with 100 rolls. The numbers of occurrences of each sum $2,3,\ldots,12$ were recorded and the cumulative distributions were plotted as shown in Figure 2.

The interval results are obtained as follows.

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[1] smallest maximum absolute error for $0 \leq r \leq 2$

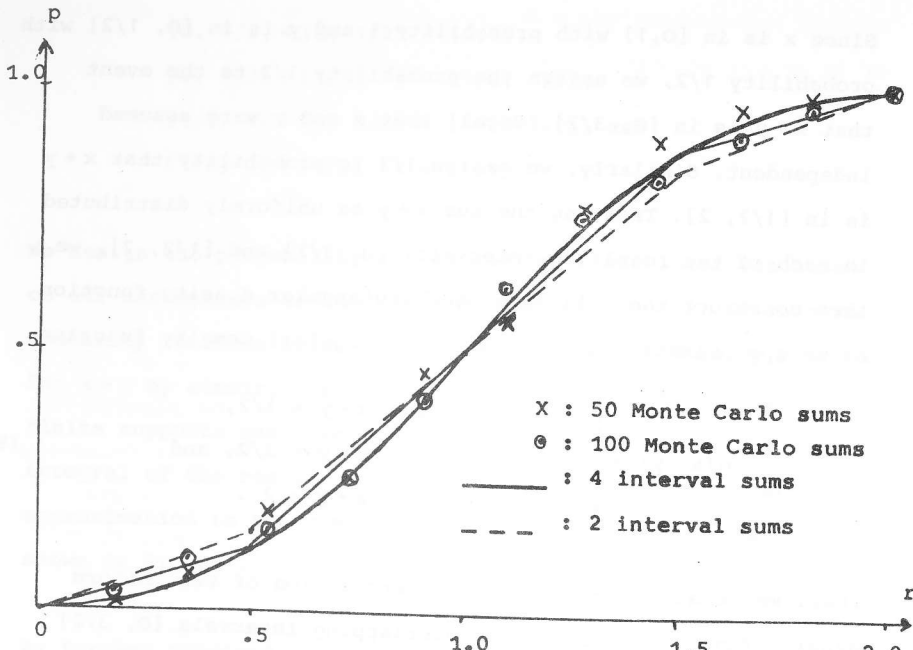

Figure 2 : A comparison of interval and Monte Carlo results in approximating the distribution of a sum (shown by the curve)

For the two-sum interval result, we subdivide the interval [0,1] for y into the two sub-intervals [0, 1/2] and [1/2, 1]. We carry out the two interval sums

$$[0,1] + [0, 1/2] = [0, 3/2], \text{ and}$$

$$[0,1] + [1/2, 1] = [1/2, 2].$$

Since x is in [0,1] with probability 1 and y is in [0, 1/2] with probability 1/2, we assign the probability 1/2 to the event that $x+y$ is in [0, 3/2]. Recall that x and y were assumed independent. Similarly, we assign 1/2 to probability that $x+y$ is in [1/2, 2]. Treating the sum $x+y$ as uniformly distributed in each of the resulting intervals, [0, 3/2] and [1/2, 2], we then construct the following step rectangular density function as an approximation to the exact (triangular) density function:

$$d(x+y) = \begin{cases} 1/3 \text{ , for } 0 \leq x+y < 1/2, \\ 2/3 \text{ , for } 1/2 \leq x+y < 3/2, \text{ and} \\ 1/3 \text{ , for } 3/2 \leq x+y \leq 2 \text{ .} \end{cases} \qquad (9)$$

Thus, we obtain $d(x+y)$ as the superposition of two uniform density functions for the two overlapping intervals [0, 3/2] and [1/2, 2].

When we integrate the density function $d(x+y)$ given above, we obtain the two-sum interval result shown in Figure 2.

Naturally, the integration of a step rectangular density function is a simple matter of assembling a continuous, piece-wise linear function.

It can be seen, from Figure 2, that the simple two-sum interval result is already of accuracy comparable to that obtained with 50 Monte Carlo sums.

For the four-sum interval result, we bisect each of the intervals for x and y and compute a density function for $x+y$ by carrying out the four sums

$$[0, 1/2] + [0, 1/2] = [0,1],$$

$$[0, 1/2] + [1/2, 1] = [1/2, 3/2],$$

$$[1/2, 1] + [0, 1/2] = [1/2, 3/2], \text{ and}$$

$$[1/2, 1] + [1/2, 1] = [1,2].$$

We assign the probability 1/4 to each of these intervals based on our assumptions on the distributions of x and y. We construct a density function which approximates the exact density function for $x+y$ by summing the four separate constant functions with finite supports over the four intervals given above. The integral of the resulting density function gives our four-sum approximation to the cumulative distribution for $x+y$ and is shown in Figure 2.

By further subdividing the intervals for x and y, we could obtain, in this way, arbitrarily accurate approximations to the exact cumulative distribution; however, the point to be made here is that, with a very small number of computations, the interval method gives results which would require much more computation using Monte Carlo methods.

We turn now to a much more interesting example involving 25 input distributions. Some are discrete density functions; some are normal; some are uniform; some are triangular; some are independent; and some are interrelated.

In Chapter VI of Reutlinger (1970), a model is proposed for a hypothetical highway project. "The model and data used ... closely approximate the model and data used in a pre-appraisal of an actual project." [loc. cit., p. 69]

For the purpose of carrying out a risk analysis, it is desired to compute the cumulative probability distribution (or an adequate approximation to it) for the ROR from the equation

$$\sum_{t=1}^{m} (1+r)^{-t}(C(t) - B(t)) = 0, \tag{10}$$

where C(t) and B(t) are, respectively, the cost and benefit for the proposed project in year t, and are defined by the following set of equations. The notation has been simplified, but the structure of the equations is the same as in Reutlinger (1970).

$$c = c_1 + c_2 + c_3 + c_4 + c_5$$

$$C(t) = \begin{cases} c/n & \text{if } t \leq n \\ 0 & \text{if } t > n \end{cases} \tag{11}$$

$$x_1(t) = (1 + e_1(t))e_2$$
$$x_2(t) = (1 + e_3(t))e_4$$

$$x_3(t) = (1 + e_5(t))e_6$$
$$x_4(t) = (1 + e_7(t))e_8$$
$$x_5(t) = (1 + e_9(t))e_{10}$$
$$x_6 = e_{11} - e_{12}$$
$$x_7 = e_{13} - e_{14}$$
$$x_8 = e_{15} - e_{16}$$
$$x_9 = e_{17} - e_{18}$$
$$x_{10} = e_{19} - e_{20}$$
$$x_{11}(t) = x_6 x_1(t)(365e_{21})$$
$$x_{12}(t) = x_7 x_2(t)(365e_{21})$$
$$x_{13}(t) = x_8 x_3(t)(365e_{21})$$
$$x_{14}(t) = x_9 x_4(t)(365e_{21})$$
$$x_{15}(t) = x_{10} x_5(t)(365e_{21})$$
$$x_{16}(t) = x_1(t)e_{11}e_{23}(365)$$
$$x_{17}(t) = x_2(t)e_{13}e_{23}(365)$$
$$x_{18}(t) = x_3(t)e_{15}e_{23}(365)$$
$$x_{19}(t) = x_4(t)e_{17}e_{23}(365)$$
$$x_{20}(t) = x_5(t)e_{19}e_{23}(365)$$
$$x_{21}(t) = x_1(t) + 2x_2(t) + 2x_3(t) + 3x_4(t) + 3x_5(t)$$
$$x_{22}(t) = e_{24} + e_{25}x_{21}(t)$$
$$x_{23}(t) = e_{26} + e_{27}x_{21}(t)$$
$$x_{24}(t) = (x_{22}(t) - x_{23}(t))e_{21}$$
$$x_{25}(t) = x_{11}(t) + \ldots + x_{15}(t)$$
$$x_{26}(t) = x_{16}(t) + \ldots + x_{20}(t)$$

$$B(t) = \begin{cases} x_{25}(t) + x_{26}(t) + x_{24}(t), & t > n, \\ 0, & t \leq n. \end{cases}$$

This is a project that takes two years (n = 2) to build a new road and will produce benefits to be measured over m - 2 years, where m is allowed to range in the analysis over the interval from 12 to 25 years. The meanings of the variable need not concern us here (material costs, labor costs, savings in road repair, savings in vehicle maintenance, etc.). What is of concern here in this example is an illustration of interval methods for constructing the cumulative probability distribution for the ROR, given estimated density functions for the input variables.

Input data are given by Reutlinger (1970) for the exogenous variables in the model as follows. Some are in the form of probability density functions; some of these are interrelated. For convenience, we give the data here in thousands of units. We use the following notation: "p = x" denotes probability x that the given value or distribution applies; "unif." denotes a uniform density function; "tri." denotes a triangular density function; "m xx, sd yy" denotes a normal density function with mean xx and standard deviation yy. With this notation, the given input data are:

$$c_1 = \begin{cases} 579 & (p = .4) \\ 466 & (p = .6) \end{cases} \qquad (12)$$

$$c_2 = \begin{cases} [150, 240] \ (\text{unif.}) \ (p = .3) \\ [240, 300] \ (\text{unif.}) \ (p = .5) \\ [300, 400] \ (\text{unif.}) \ (p = .2) \end{cases} \text{if } c_1 = 579$$

$$\begin{cases} [200, 300] \ (\text{unif.}) \ (p = .3) \\ [300, 340] \ (\text{unif.}) \ (p = .5) \\ [340, 440] \ (\text{unif.}) \ (p = .2) \end{cases} \text{if } c_1 = 466$$

c_3 = [46.2, 92.4] (unif.)

c_4 = [10, 30] (tri.)

c_5 = [941.85, 1163.5] (tri.)

n = 2

e_1 = [.04, .08] (unif.)

e_2 = m 41, sd 3.35

e_3 = [.06, .10] (unif.)

e_4 = [15, 35] (tri.)

e_5 = [.04, .08] (unif.)

e_6 = m 6, sd 1

e_7 = e_3 ("fully correlated")

e_8 = m 15, sd 3.3

$$e_9(t) = \begin{cases} 0, & \text{if } t \leq 4 \\ -.2, & \text{if } t > 4 \end{cases}$$

$$e_{10} = \begin{cases} 35 & (p = .25) \\ 0 & (p = .75) \end{cases}$$

e_{11} = .0613 (1 + y), y = [-.12, .15] (unif.)

e_{12} = .0479

e_{13} = .1076 (1 + y)(1 + h), h = [-.05, .10] (unif.)

e_{14} = .0670 (1 + h)

e_{15} = .1516 (1 + y)

e_{16} = .1034

e_{17} = .215 (1 + y)

e_{18} = .141

e_{19} = .215 (1 + y)

e_{20} = .140

e_{21} = 64

e_{22} does not occur and can be omitted

e_{23} = 0

e_{24} = 417

e_{25} = [3, 5] (unif.)

e_{26} = 600

e_{27} = [1, 2.5] (unif.)

m = [12, 25] (tri.)

The intervals shown in the above list are the ranges of values (from low to high) which are assumed for the variables concerned. Thus it is given in the model that c_3 is uniformly distributed in the interval [46.2, 92.4]; therefore we can assign the probability 1.0 to the interval [46.2, 92.4] for c_3. Similarly, c_3 is in the first half of the interval, namely [46.2, 69.3] with probability 1/2. In fact, we can assign probability p to any subinterval (of the given interval for c_3) which has width p times the width of the given interval.

For those variables with triangular density functions, such as c_4, c_5, e_4, etc., we can assign probabilities to subintervals of the given interval supports (range of values) by using the mean value of the triangular density function over the subinterval

times the width of the subinterval. We can do the same for variables with normal density functions or any other given density functions. The step rectangular approximations we construct in this way will integrate to 1.0 over the range of values of the variable in question. For density functions such as the normal distribution with infinite "tails", we can chop off the tails in such a way that the remaining finite support contains most of the density function (so that the remaining area is as close to 1.0 as we like). Alternatively, we can approximate a normal density function by a step rectangular one with finite support (say [m - 2sd, m + 2sd]) whose intergral is 1.0.

In light of the above discussion, we can find an interval for each of the density functions given in (12). Substituting these intervals into the expressions on the right hand sides of the formulas in (11), we can, using interval arithmetic, find intervals containing the values of C(t) - B(t) for t from 1 to m. Using the resulting interval coefficients in (10), we can find an interval containing the range of values of ROR.

Assuming a uniform density function for ROR in the interval found in the way just described, we would obtain a linear cumulative distribution for ROR as a first approximation.

By choosing a subinterval from each of the finite supports of each of the independent density functions in (12), we can again obtain a corresponding interval of values for ROR and can assign a probability resulting from multiplying the probabilities

associated with the chosen subintervals for the input density functions. If we do this for a number of vectors of input intervals, the resulting intervals for ROR may be overlapping. We can construct a density function for ROR from these in a way similar to the constructions in the previous examples. Thus, we can refine the first approximation (obtained using the entire interval for each input variable) by subdivision of selected input intervals.

A good strategy is to subdivide those intervals first which will most affect the final resulting distribution for ROR, and stop the process when the resulting cumulative distribution for ROR changes by less than some pre-determined mesure.

We can rank-order the input variables ($c_1, c_2, \ldots, e_1, \ldots, e_{27}$, m in the example) with respect to their influence on the value of ROR using elasticities. The elasticity of ROR with respect to a given input variable is defined as the ratio of percentage variations in ROR to the percentage variations in the input variable. This definition is not precise, of course, since a value for elasticity obtained in this way would depend on the magnitude of the variations in the input variable. In the limit as these become small, the elasticity is the logarithmic derivative of ROR with respect to an input variable, VAR, namely:

$$(dROR/ROR)/(dVAR/VAR).$$

However, it is not necessary to take derivatives in practise. Instead, we may simply compute the interval of values of ROR

corresponding to an interval of values of a given input variable, while setting other input variables at nominal real values. Then we can take the elasticity to be

$$(w(ROR)/m(ROR))/(w(VAR)/m(VAR)), \qquad (13$$

where w(ROR) denotes the width of the interval ROR and m(ROR) denotes the midpoint of the interval ROR.

Such a rank ordering requires one interval solution for ROR for each independent input density function. As we will see, this means 17 such computations in the example under discussion. A program for computing elasticities is included in the appendix, as part of the program for finding an interval of values for ROR.

We now describe the results obtained using the methods discussed in this section for the model (11) for the data given by (12). In particular, we will discuss how we obtained a probability distribution for ROR. The results were obtained using the interactive computer program given in the appendix and can be verified on any computer with a FORTRAN compiler.

For the nominal values of the input (given by statements 3. to 37. in the program) we obtained the value

$$ROR = [-.0338, -.0338] \quad (RL = RU = -.0338).$$

This figure, which was independently verified by hand computation, lies outside the distribution given in Reutlinger (1970). We conclude that the results given in Reutlinger (1970) are

incorrect[2]. There are some obvious errors in the paper cited. In particular, the equations for C(t) and $x_{24}(t)$ should read as given in (11). Nevertheless, it is an interesting example.

From the data given in (12), we identify 18 independent density functions:

D_1, a discrete density function for the coefficient c_1, with probability .4 assigned to the value 579,000 and p = .6 assigned to the value 466,000;

D_2, a density function for c_2, while dependent on D_1, it does involve an additional independent distribution for each of the values of c_1;

D_3, a uniform density for c_3;

D_4, a triangular density for c_4;

D_5, a triangular density for c_5;

D_6, a uniform density for e_1;

D_7, a normal density for e_2;

D_8, uniform for e_3;

D_9, triangular for e_4;

D_{10}, uniform for e_5;

D_{11}, normal for e_6;

D_{12}, normal for e_8;

---

[2] Or, perhaps, there are some typographical errors in the data given.

D_{13}, uniform for the variable we have called y in (12), this affects the densities for e_{11}, e_{13}, e_{15}, e_{17}, and e_{19};

D_{14}, uniform for the variable we have called h in (12), this affects the densities for e_{13} and e_{14};

D_{15}, discrete for e_{10} as shown in (12);

D_{16}, uniform for e_{25};

D_{17}, uniform for e_{27};

D_{18}, triangular for m.

If we truncate the three normal density functions, D_7, D_{11}, and D_{12} at two standard deviations from their means with appropriate elevation of the values of these density functions so that the integrals are 1.0 over the finite supports, then we can find intervals of support for each of the above densities as follows: D_1 in [466,000., 579,000.]; D_2 in [150,000., 440,000.]; D_3 in [46,200., 92,400.]; D_4 in [10,000., 30,000.]; D_5 in [941,850., 1,163,500.]; D_6 in [.04, .08]; D_7 in [34.3, 47.7]; D_8 in [.06, .10]; D_9 in [15, 35]; D_{10} in [.04, .08]; D_{11} in [4, 8]; D_{12} in [8.4, 21.6]; D_{13} in [-.12, +.15]; D_{14} in [-.05, +.10]; D_{15} in [0, 35]; D_{16} in [3, 5]; D_{17} in [1, 2.5]; D_{18} in [12, 25].

Note that e_{11}, e_{13}, e_{14}, e_{15}, e_{17}, and e_{19} have densities correlated with those of y and h (see (12)). We will explain what we did about this in Section 3 on interrelated distributions.

The distribution of m in [12, 25] (see (12)) has to be treated in a different way from the other densities because of the way that m enters into the computation of ROR; see equation (10).

Using the nominal value m = 20, we carried out a single interval solution for ROR, starting with the full interval ranges for the 17 density functions $D_1,\ldots,D_{17}$, and obtained the interval [-.146, .159] containing the range of possible values for ROR when m = 20. For this and other values of m, we obtained the results shown in Figure 3. Thus, for m in the interval [12, 25], we obtain the interval [-.290, .166] containing the range of possible values of ROR under all the given conditions. Also shown in Figure 3 are two discrete approximations to the density function for m which can be used in the final construction of the cumulative distribution for ROR. There are, of course, many other possibilities for approximating the density function for m, using the data contained in Figure 3.

Next, we used the program (again with m = 20) to find the elasticity of ROR with respect to each of the 17 variables with distributions $D_1,\ldots,D_{17}$. This was carried out for the nominal input. Since ROR turned out to be negative for the nominal input, the elasticities come out negative, using the definition (13). We use the absolute value of these for our rank ordering of the influence of the variables on the value of ROR. The absolute values of the elasticities found were as follows (given as pairs (I, |E(I)|), where I is an index corresponding to D_I): (1, .65),

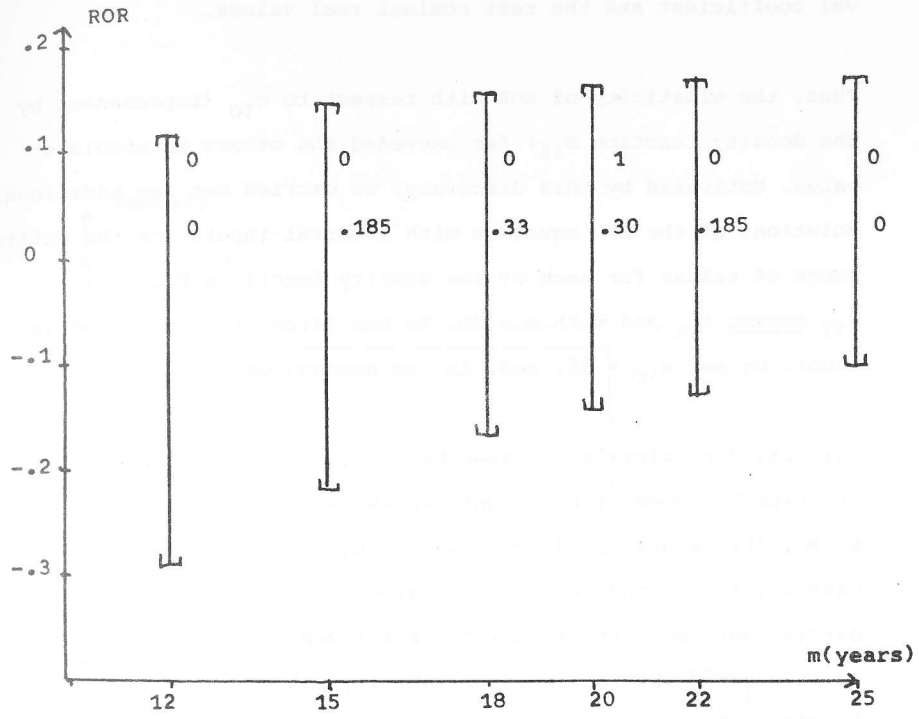

Figure 3 : Intervals containing the distributions of ROR for various values of m, together with two discrete approximations to the triangular density function for m

(2, .43), (3, .10), (4, .027), (5, 1.4), (6, .035), (7, .62), (8, .16), (9, 1.1), (10, .017), (11, .29), (12, 1.1), (13, .12), (14, .021), (15, 5.6), (16, 1.6), (17, .71). Each of these elasticities required one interval solution for ROR with one interval coefficient and the rest nominal real values.

Thus, the elasticity of ROR with respect to e_{10} (represented by the density function D_{15}) far exceeded the others in absolute value. Motivated by this discovery, we carried out <u>two</u> additional solutions of the ROR equation with interval inputs for the entire range of values for each of the density functions D_1, ..., D_{17} <u>expect</u> D_{15} and with m = 20. In the first of these computations, we set e_{10} = 35, and, in the second, we set e_{10} = 0.

This was done since e_{10}, given by (12), has a discrete density function D_{15} consisting of only those two values. If e_{10} had been given, say, a uniform density function, then we would, instead, have subdivided the interval of values into two subintervals and carried out two interval solutions for ROR using each of those. As it was, we then assigned probability .25 to the ROR result [-.0744, .1590] using e_{10} = 35; and we assigned the probability .75 to the ROR result [-.146, .0956] using e_{10} = 0.

When we superpose the two intervals given above with the assigned probabilities, we obtain the density function for ROR shown by the solid lines in Figure 4. The integral of this density function is the cumulative distribution for ROR shown in Figure 5 by the solid lines.

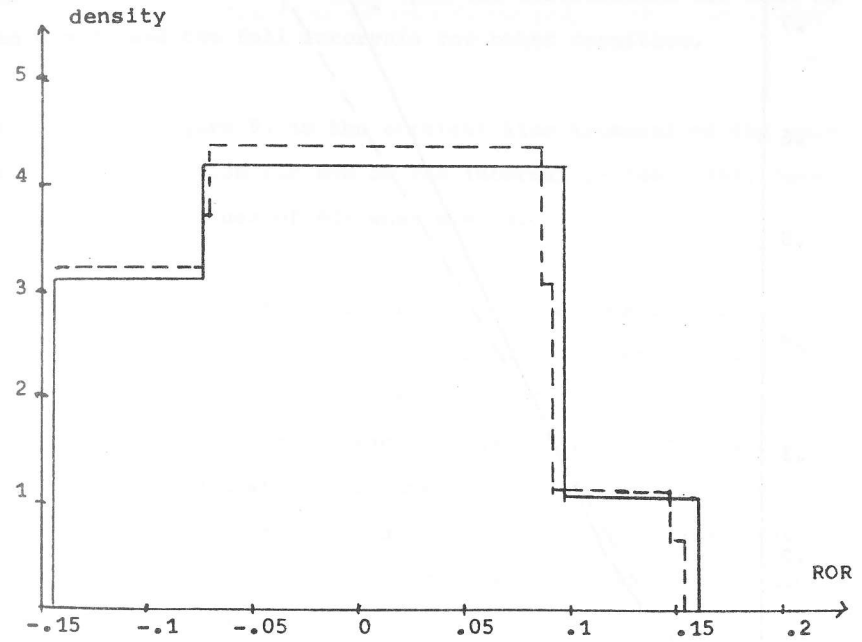

Figure 4 : Step rectangular density functions for ROR

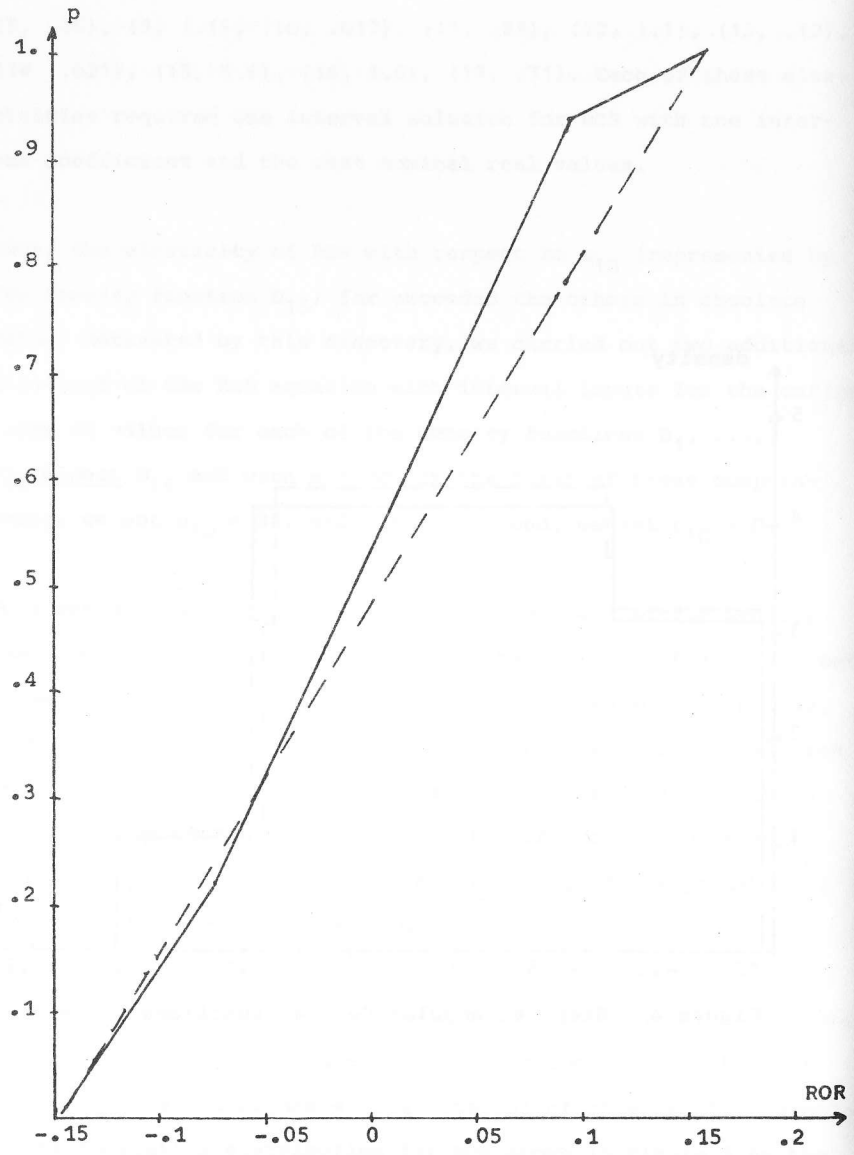

Figure 5 : Cumulative probability distribution for ROR

In Figure 4, there is an additional density function shown by the dashes lines. It was obtained by carrying out four more solutions for ROR corresponding to the four possible combinations of values for e_{10} and c_1 (with c_2 chosen accordingly, c_2 = [150,000., 400,000.] if c_1 = 579,000. and c_2 = [200,000., 440,000.] if c_1 = 466,000.). We include this result here to show the small magnitude of the effect of details of the density functions for c_1 and c_2 upon the distribution for ROR. We used m = 20 and the full intervals for other densities.

Also shown in Figure 5, is the straight line integral of the uniform density function for ROR on the interval [-.146, .159] containing all the values of ROR when m = 20.

Further subdivisions of the intervals for the remaining density functions will produce smaller effects than notices in the difference between the two graphs in Figure 5. Thus, for example, we carried out a subdivision of each of the density functions D_{16} and D_5 (which were next in the rank ordering of elasticities). This required four more interval solutions of the ROR equation and did not produce a significant change in the density function for ROR shown in Figure 4.

The effect of varying the approximation to the triangular density function for m (see Figure 3) is also small.

We conclude that the cumulative distribution for ROR shown in Figure 5 adequately reflects the information contained in the model (11) and the data (12). We computed ROR by interval methods, in all, 34 times to obtain these results (one each to obtain the 17 elasticities, once for the nominal data, 4 times to analyze the effects of e_{10} and c_1, twice each for D_{15}, D_{16}, and D_5, and 6 times for m).

The Monte Carlo method would require many hundreds of solutions of the ROR equation to obtain comparable results [Reutlinger (1970)].

We can illustrate the idea of "risk analysis" using the cumulative distribution for ROR shown in Figure 5. It would seem from the figure that the proposed highway project s indeed a "risky" proposition. The probability is greater than 1/2 that the project will not even break even in 20 years. There is, however, a probability of about .25 = (1.0 - .75) that the proposed project will yield a rate of-return of at least 5% over 20 years. From the graph in Figure 5, we can deduce a variety of such estimates. Thus, the probability that the rate-of-return will exceed 7.5% over twenty years is about .15. By studying such a graph, we can see a measure of the risks involved in a proposed project in the sense of a relationship between lower bounds (or upper bounds) on possible rates-of-return and estimated probabilities that these will occur.

3. Interrelated distributions

In the example discussed in the previous section concerning a model for a highway project, there are several interrelated distributions among those given for the exogenous variables. In particular, those for e_{11}, e_{13}, e_{15}, e_{17}, and e_{19} are interrelated as are those for e_{13} and e_{14} (and $e_7 = e_3$).

In this section we will discuss how to best handle such interrelated distributions using interval methods.

First, consider the equation for x_7, namely

$$x_7 = e_{13} - e_{14} .$$

If we use interval computation in a naive way, we obtain, from the defining equations for e_{13} and e_{14},

$$e_{13} = .1076\ (1 + [-.12, .15])(1 + [-.05, .10])$$

$$= [.0899536, .136114] \text{ , and}$$

$$e_{14} = .0670\ (1 + [-.05, .10])$$

$$= [.06365, .0737] \quad ; \text{ and then we obtain}$$

$$x_7 = [.0162536, .072464].$$

What is wrong with this computation is that we have not made use of the information that e_{13} and e_{14} are interrelated. In order to make use of this information, we must proceed as follows. We form an expression for $e_{13} - e_{14}$ which contains the information concerning the interrelation, namely

$$x_7 = e_{13} - e_{14} = (.1076\ (1 + y) - .0670)\ (1 + h).$$

In this way, the independent variables y and h each occur only once in the expression and the resulting interval computation will produce the exact range of values for x_7, namely

$$x_7 = (.1076\ (1 + [-.12, .15]) - .067)(1 + [-.05, .1])$$
$$= [.0263036, .062414].$$

The naive use of interval arithmetic has over-estimated the exact range here. The repeated naive use of interval arithmetic can lead to useless results.

In order to make use of the information concerning interrelated probability distributions during interval computation, we must preserve the identity of the variables until the final arithmetic computation.

Thus, in the example at hand, we must do even more than factor out the term (1 + h) from the difference $e_{13} - e_{14}$. We must go all the way to the quantity of interest. Here, that is the rate-of-return, ROR. We must compute the coefficients C(t) - B(t) for t = 1, ..., 1 using all the information available concerning the interrelation of the distributions for the exogenous variables entering into the computation of these coefficients. In this way, a subdivision of the interval for y, namely [-.12, .15], if it needs to be done, will correctly affect the final distribution for r.

In the example under discussion, the term $x_{26}(t)$, which occurs in the defining equation for B(t), involves the five interrelated distributions for e_{11}, e_{13}, e_{15}, e_{17}, and e_{19}. We can factor out

the term (1 + y) and express

$$x_{26}(t) = e_{23}\ 365(x_1\ .0613 + x_2\ .1076(1+h) + x_3\ .1516 + x_4\ .215 + x_5\ .215)(1+y).$$

In this way, the interrelations among the five distributions will be automatically taken care of in assigning probabilities to sub-intervals of the range of values of $x_{26}(t)$. Actually, from the data given, since $e_{23} = 0$, we would obtain $x_{26}(t) = 0$. For other values of e_{23}, however, the variable x_{26} would definitely come into play. Then again, for the given data the values of x_1, x_2, x_3, x_4, x_5 are always positive and it follows from properties of interval arithmetic that the sum shown above for $x_{26}(t)$ it in this case distributive. Thus, we can compute the exact range of values of $x_{26}(t)$ using interval arithmetic even without first factoring out the term (1 + y). This was not the case for the difference $e_{13} - e_{14}$ discussed earlier.

In any case, the interrelated distributions for e_{11}, e_{13}, e_{15}, e_{17}, and e_{19} should be dealt with computationally as generated by the two independent distributions for y and h alone. Thus, in an interval computation, if we subdivide the interval [-.12, .15] for y, we assign resulting probabilities simultaneously to e_{11}, e_{13}, e_{15}, e_{17}, and e_{19}. In this way, we can determine correctly the probability to assign to a sub-interval of the range of possible values of the variable $x_{26}(t)$.

Appendix

We list here an interactive program in FORTRAN (with commentary) for carrying out the interval method just described on the example we have been discussing.

Since we will generate up to 25 coefficients for costs and 25 for benefits and since we want intervals for each of these, we will create four arrays of dimension 25: CL , CU , BL , BU . The notation consists of mnemonics for "cost lower", "cost upper", "benefit lower", and "benefit upper". An element of one of these arrays will be, for example, CL(2): the estimated lower bound on the cost in year 2.

We will need arrays for the input data in order to enter intervals of possible values for each of 17 independent distributions as discribed previously. These will be denoted by DL and DU for "D lower" and "D upper". An interval of values for one of the input variables, say the one with density function D_3, is entered as a value of DL(3) and a value of DU(3). Thus, if we enter DL(3) = 46200. and DU(3) = 92400., then the interval [DL(3), DU(3)] contains the value of c_3 with probability 1. The distribution for m, namely D_{18}, will be treated in another way.

Since we wish to illustrate the usefulness of "elasticities" we create another array of dimension 17, denoted by E.

Thus the first statement in our FORTRAN program is

```
1.    DIMENSION CL(25), CU(25), BL(25), BU(25), DL(17), DU(17), E(17)
```

Since the ranges of values of the input are rather wide, we decided to use partial double precision during the computation of the ROR. For this we need the next statement in our program

```
2.    DOUBLE PRECISION PL, PU, PPL, PPU
```

The four variables indicated are the lower and upper values of the polynomial and their derivatives (P Prime, Lower and Upper).

Next, as an initialisation procedure, we enter nominal values for the input (Reutlinger's "single value estimates")

```
3.    M=20
4.    DL(1)=466000.
5.    DU(1)=466000.
6.    DL(2)=311150.
7.    DU(2)=311150.
8.    DL(3)=92400.
9.    DU(3)=92400.
10.   DL(4)=15000.
11.   DU(4)=15000.
12.   DL(5)=1023000.
13.   DU(5)=1023000.
14.   DL(6)=.06
15.   DU(6)=.06
```

```
16.     DL(7)=41.
17.     DU(7)=41.
18.     DL(8)=.08
19.     DU(8)=.08
20.     DL(9)=23.
21.     DU(9)=23.
22.     DL(10)=.06
23.     DU(10)=.06
24.     DL(11)=6.
25.     DU(11)=6.
26.     DL(12)=15.
27.     DU(12)=15.
28.     DL(13)=0.
29.     DU(13)=0.
30.     DL(14)=0.
31.     DU(14)=0.
32.     DL(15)=0.
33.     DU(15)=0.
34.     DL(16)=3.6
35.     DU(16)=3.6
36.     DL(17)=2.25
37.     DU(17)=2.25
```

At this point in the program, we want to go to the computation of the coefficients for the ROR equation. We will enter this same point later with different input. We write

```
38.     GO TO 50
```


After the coefficients have been computed, we will have filled the arrays CL, CU, BL, and BU. Then we will be in a position to compute values of the polynomial and its derivative--in fact lower and upper bounds to these, using interval methods. Thus, we will return from the coefficient computation, beginning at 50, to

```
39.     5 K=0
40.       RL=.2
41.       RU=.2
```

These instructions initialize the iteration count for the Newton method we will use to 0, and set the initial values for the lower and upper values of ROR to .2 (this seemed a reasonable initial guess).

Next, we evaluate the polynomial and its derivative at both our current estimates of the lower (RL) and upper (RU) bounds on the ROR. This is done recursively by

```
42.     10 PL=CL(1)-BU(1)
43.        PU=CU(1)-BL(1)
44.        PPL=0.
45.        PPU=0.
46.        DO 20 I=2,M
47.        PPL=PL+(1.+RU)*PPL
48.        PPU=PU+(1.+RL)*PPU
49.        PL=CL(I)-BU(I)+(1.+RU)*PL
50.     20 PU=CU(I)-BL(I)+(1.+RL)*PU
```

The derivation of the above algorithm is given in Chapter 9 of Moore (1979); as is the derivation of the next several statements in the program

```
51.       R1 = RL-PU/PPU
52.       R2 = RU-PL/PPL
53.       IF(K.GE.2.AND.R1.GE.RL) GO TO 30
54.       RL=R1
55.       RU=R2
56.       WRITE(6,31)K,RL,RU,PL,PU,PPL,PPU
57.       K=K+1
58.       IF(K.GE.15) GO TO 40
59.       GO TO 10
```

The previous nine statements in the program consist of the Newton algorithm for finding the ROR (lower and upper)--with some exits based upon a reasonable convergence criterion (statement 53.) and a cut-off to prevent run-away iteration in case of divergence (statement 58.). The write statement at 56. was put in to study the convergence of the algorithm. It is left in here because it does no harm to look at some intermediate results. Occasionally, it is of great help. Following the determination of the ROR interval, we proceed with

```
60.    30 WRITE(6,31)K,RL,RU,PL,PU,PPL,PPU
61.    31 FORMAT(1X,I2,6E11.3)
62.       GO TO 300
```


This writes the final results of the ROR computation and then goes to the interactive part of the program, asking for other input for more cases.

```
63.    40 WRITE(6,41)K,RL,RU,PL,PU
64.    41 FORMAT(1X,'DIVERGES? K=  ', I2, 4E11.3)
65.       GO TO  300
```

Statements 63. and 64. write the results of the 15th iteration for the ROR in case the convergence criteria have not been met. Control is then passed to the interactive part of the program.

The next set of instructions consists of the interactive part of the program. There are entry points at 200, 103, and 300.

```
66.   200 WRITE(6,100)
67.   100 FORMAT(1X,'M= ? TYPE IN COLS 1,2')
68.       READ(5,101) M
69.   101 FORMAT(I2)
70.   103 WRITE(6,102)
71.   102 FORMAT(1X,'TO CHANGE D(I) TYPE I COLS 1,2')
72.       READ(5,101) I
73.       WRITE(6,104)
74.   104 FORMAT(1X,'TYPE VALUE OF DL(I) DECIMAL FORM COLS 1-20')
75.       READ(5,105) DL(I)
76.   105 FORMAT(F20.4)
77.       WRITE(6,106)
78.   106 FORMAT(1X,'TYPE DU(I) VALUE')
79.       READ(5,105) DU(I)
```

```
80.    300 WRITE(6,107)
81.    107 FORMAT(1X,'TYPE 00 TO RERUN, 01 TO STOP, 02 TO CHANGE M,
82.       c03 TO CHANGE SOME D, 04 TO PRINT ELASTICITIES')
```

Note: the c here denotes a continuation of the previous statement.

```
83.         READ(5,101) J
84.         IF(J.EQ.0) GO TO 50
85.         IF(J.EQ.2) GO TO 200
86.         IF(J.EQ.3) GO TO 103
87.         IF(J.EQ.4) GO TO 70
88.         STOP
```

This interactive part of the program allows us to rerun the computation of an interval of values for ROR with new sets of intervals for input variables with distributions (except M is entered as an integer).

Note that, after we have changed whatever input we wish, control is passed to 50, where the computation of the cost and benefit coefficients is carried out. We list this section of the program next.

```
89.     50 DO 60 I = 1,M
90.        IF(I.GT.2) GO TO 51
91.        CL(I)=DL(1)+DL(2)+DL(3)+DL(4)+DL(5)
92.        CU(I)=DU(1)+DU(2)+DU(3)+DU(4)+DU(5)
93.        CL(I)=CL(I)/2.
94.        CU(I)=CU(I)/2.
95.        BL(I)=0.
96.        BU(I)=0.
97.        IF(I.LE.2) GO TO 60
```


This instructions compute the intervals of values for costs and benefits during the first two years. After that, costs become zero and benefits become positive.

```
98.     51 X1L=(1.+DL(6))*DL(7)
99.        X1U=(1.+DU(6))*DU(7)
100.       CL(I)=0.
101.       CU(I)=0.
102.       X2L=(1.+DL(8))*DL(9)
103.       X2U=(1.+DU(8))*DU(9)
104.       X3L=(1.+DL(10))*DL(11)
105.       X3U=(1.+DU(10))*DU(11)
106.       X4L=(1.+DL(8))*DL(12)
107.       X4U=(1.+DU(8))*DU(12)
108.       X5L=DL(15)
109.       X5U=DU(15)
110.       IF(I.LE.4) GO TO 52
111.       X5L= .8*X5L
112.       X5U= .8*X5U
113.    52 X6L=.0134+.0613*DL(13)
114.       X6U=.0134+.0613*DU(13)
115.       X7L=(.1076*(1.+DL(13))-.0670)*(1.+DL(14))
116.       X7U=(.1076*(1.+DU(13))-.0670)*(1.+DU(14))
117.       X8L=.1516*(1.+DL(13))-.1034
118.       X8U=.1516*(1.+DU(13))-.1034
119.       X9L=.215*(1.+DL(13))-.141
120.       X9U=.215*(1.+DU(13))-.141
121.       X10L=X9L + .001
122.       X10U=X9U + .001
```

```
123.      A = 2336O.
124.      QL=64.*(DL(16)-DU(17))
125.      QU=64.*(DU(16)-DL(17))
126.      A1L=QL+A*X6L
127.      A1U=QU+A*X6U
128.      A2L=2.*QL+A*X7L
129.      A2U=2.*QU+A*X7U
13O.      A3L=2.*QL+A*X8L
131.      A3U=2.*QU+A*X8U
132.      A4L=3.*QL+A*X9L
133.      A4U=3.*QU+A*X9U
134.      A5L=3.*QL+A*X1OL
135.      A5U=3.*QU+A*X1OU
```

To this point the interval computation of the ranges of coefficients is straightforward. It can be checked from the given maximum ranges of the input distributions that no sign changes occur in the intervals involved above. However, for the next five intervals, we must use the general formula for interval multiplication since sign changes do occur in some of the intervals involved.

```
136.      B1L=AMIN1(A1L*X1L,A1L*X1U,A1U*X1L,A1U*X1U)
137.      B1U=AMAX1(A1L*X1L,A1L*X1U,A1U*X1L,A1U*X1U)
138.      B2L=AMIN1(A2L*X2L,A2L*X2U,A2U*X2L,A2U*X2U)
139.      B2U=AMAX1(A2L*X2L,A2L*X2U,A2U*X2L,A2U*X2U)
14O.      B3L=AMIN1(A3L*X3L,A3L*X3U,A3U*X3L,A3U*X3U)
141.      B3U=AMAX1(A3L*X3L,A3L*X3U,A3U*X3L,A3U*X3U)
142.      B4L=AMIN1(A4L*X4L,A4L*X4U,A4U*X4L,A4U*X4U)
143.      B4U=AMAX1(A4L*X4L,A4L*X4U,A4U*X4L,A4U*X4U)
```



```
144.      B5L=AMIN1(A5L*X5L,A5L*X5U,A5U*X5L,A5U*X5U)
145.      B5U=AMAX1(A5L*X5L,A5L*X5U,A5U*X5L,A5U*X5U)
```

We are now ready to compute the lower and upper limits of benefit in the year I corresponding to the given input intervals.

```
146.      BL(I)= -11712. +B1L+B2L+B3L+B4L+B5L
147.      BU(I)= -11712. +B1U+B2U+B3U+B4U+B5U
148.   6O CONTINUE
149.      GO TO 5
```

This completes the computation of the coefficients for the ROR equation. Some algebraic manipulations of the formulas were carried out prior to programming in order to obtain the best form for interval computation. Multiple occurrences of a variable were eliminated wherever possible. Little or no excess width remains.

The remainder of the program is a computation of the elasticities which was added here.

```
15O.   7O RR=(RU-RL)/(.5*(RU+RL))
151.      DO 8O I=1,17
152.      DEL=(DU(I)-DL(I))/(.5*(DU(I)+DL(I))
153.      IF(ABS(DEL).LE.1.E-5)GO TO 8O
154.      E(I)=RR/DEL
155.      WRITE(6,71)I,E(I)
156.   71 FORMAT(1X,'ELASTICITY', 3X,I2,E1O.2)
157.   8O CONTINUE
158.      END
```

The form of the program for elasticities allows us to check that w have only put in one interval of values for the input. We should g only one elasticity printed out.

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INTERVALLMATHEMATISCHE BEHANDLUNG VON ANFANGSWERTAUFGABEN GEWÖHNLICHER DIFFERENTIALGLEICHUNGEN

von

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Es werden in dieser Arbeit Anfangswertaufgaben (AWAn) bei Systemen gewöhnlicher Differentialgleichung 1.Ordnung mit Intervallkoeffizienten und Intervallanfangswerten betrachtet. Jede dieser Intervallanfangswertaufgaben (IAWAn) wird als Menge reeller AWAn aufgefaßt:

$$\{\dot{x} = f(t,c,x(t)),\ x(\underline{t}) = x_0 \in [x_0] \in V_n(I(R))$$
$$c \in [c] \in V_m(I(R)),\ t \in T = [\underline{t},\overline{t}] \in I(R)\} \qquad (1)$$
$$\text{mit } x: T \to R^n,\ f: T \times [c] \times R^n \to R^n.$$

($I(R)$ - Menge der reellen abgeschlossenen, beschränkten Intervalle, $V_k(I(R))$ - Menge der k-dimensionalen Intervallvektoren.) Die Menge wird erzeugt über dem Parametervektor $c \in [c]$ und dem Vektor der Anfangsbedingungen $x_0 \in [x_0]$. Zur Sicherung von Existenz und Eindeutigkeit wird vorausgesetzt, daß für jedes $c \in [c]$ die Funktion f die Voraussetzungen des bekannten Satzes von PICARD-LINDELÖF erfüllt. Eine Funktion x gehört zur Lösungs-

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menge $\{\hat{x}\}$ der IAWA, wenn

1. $\exists\, c \in [c]$, so daß $\dot{x}(t) = f(t,c,x(t))$ auf T gilt,
2. $\exists\, x_0 \in [x_0]$, so daß $\dot{x}(\underline{t}) = x_0$ gilt.

Die Menge $\{\hat{x}\}$ läßt sich im allgemeinen nicht auf einfache Weise beschreiben. In der Intervallmathematik werden Intervalloberhüllen für $\{\hat{x}\}$ gesucht, d.h. es wird versucht $\{\hat{x}\}$ durch eine intervallwertige Funktion einzuschließen. Aber auch eine solche Intervalloberhülle oder gar die Intervallhülle $[\hat{x}]$ (optimale Intervalleinschließung) sind nicht leicht bestimmbar. In speziellen Fällen läßt sich unter Monotonievoraussetzungen an f die Intervallhülle (theoretisch) a priori angeben (man vgl. [3]). Im folgenden sollen keine derartigen Monotonieforderungen an das Problem gestellt werden. Die Untersuchung solcher IAWAn ist aus folgenden Gründen von Bedeutung.

1. In Naturwissenschaft und Technik treten AWAn auf, bei denen sowohl Koeffizienten in den Differentialgleichungen als auch Anfangswerte aus Messungen hervorgegangen und damit fehlerbehaftet sind, und es wird nach dem Einfluß dieser Fehler auf die Lösung gefragt. Faßt man die fehlerbehafteten Daten als Intervallgrößen auf, so kann die Frage durch Betrachten und Lösen von IAWAn beantwortet werden.
2. Eine Reihe von komplexen numerischen Aufgabenstellungen enthalten als Teilprobleme AWAn, z.B. Probleme der optimalen Steuerung mit konzentrierten Parametern; eine Reihe von numerischen Aufgabenstellungen werden in äquivalente AWAn überführt, z.B. nichtlineare Gleichungssysteme mit dem

DAVIDENKO-Einbettungstechnik. Auch in diesen Fällen sind häufig Fehlerabschätzungen gefragt. (Zur intervallmathematischen Behandlung fehlerbehafteter Steuerprobleme vgl. man [4]).

In den vergangenen Jahren wurde eine größere Anzahl von Verfahren zur intervallmathematischen Behandlung von AWAn ohne Intervallkoeffizienten und Intervallanfangswerte veröffentlicht. Mit solchen Verfahren werden nur Verfahrensfehler und Rundungsfehler erfaßt und eingeschlossen. Diese Verfahren lassen sich meist nach gewisser Modifizierung auch auf AWAn mit Intervallanfangswerten anwenden. Einige Arbeiten berücksichtigen von vornherein Intervallanfangswerte, z.B. [1] und [7]. Der allgemeinen Aufgabenstellung widmen sich die Veröffentlichungen [5] und [6], in denen zwei Verfahren angegeben werden. Die charakteristische Vorgehensweise solcher Verfahren für allgemeine IAWAn wird am Beispiel des folgenden Verfahrens 1.Ordnung deutlich. Betrachtet wird die Problemstellung (1) mit

$$f(t,c,x(t)) = f(c,x(t)),\ [x_0] = [\underline{x}_0,\overline{x}_0],\ \underline{t} = 0.$$

Es sei eine a-priori-Abschätzung $[a] = [\underline{a},\overline{a}]$ für $\{\hat{x}\}$ bekannt mit

$$\underline{a}_i < \underline{x}_{0i} \leq \overline{x}_{0i} < \overline{a}_i,\ i = 1,2,\ldots,n; \qquad (2)$$

weiterhin sollen Zahlen $l_i > 0$ existieren, so daß

$$d(F_i([c],[x])) = l_i d([x]),\ i = 1,2,\ldots,n,$$

für alle $[x] \subseteq [a]$ gilt. ($[f] = (F_i)$ Intervallerweiterung von f). Wegen (2) existiert ein $h > 0$, so daß für alle $t \in [0,h]$ gilt,

$$[x_0] + t\left[f([c]), [a])\right] \subset [a].$$

Das Intervall [0,h] wird in p (p ∈ N) Teilintervalle

$$T_s = \left[\frac{s-1}{p}h\,,\ \frac{s}{p}h\right]\ ,\ s = 1,2,\ldots,p,$$

zerlegt. Für jedes p wird für alle t ∈ [0,h] eine intervallwertige Funktion $[x_p]$ folgendermaßen definiert:

$$[x_p(0)] = [x_0],$$

für $t_s = \frac{s}{p}h$: $[x_p(t_s)] = [x_p(t_{s-1})] + \frac{h}{p}[f([c],[z_s])]$

mit

$$[z_s] = [x_p(t_{s-1})] + [0,\tfrac{h}{p}][f([c], [a])],$$

für

$$t_{s-1} < t < t_s\colon [x_p(t)] = [x_p(t_{s-1})] + (t-t_{s-1})[f([c],[z_s])]\ .$$

Für die Funktion $[x_p]$ gilt u.a. $x(t) \in [x_p(t)]$ für jedes $x \in \{\hat{x}\}$. Die Oberhüllenfunktion $[x_p]$ kann außerhalb von [0,h] fortgesetzt werden.

RAITH beschreibt in [7] ein Iterationsverfahren zur Einschließung der Lösungsmenge des Problems (1) für den Fall d([c]) = 0. An die Funktion f der AWA wird keine Monotonievoraussetzung gestellt. Es wird nun eine analoge Vorgehensweise gewählt, um die Lösungsmenge $\{\hat{x}\}$ des Problems (1) einzuschließen, wobei aber die Klasse der Funktion f eingeschränkt werden muß auf Polynome in x und t. Das

Verfahren wird unterteilt in den Basisschritt (Berechnung einer Anfangseinschließung von $\{\hat{x}\}$) und eine Folge von Verbesserungsschritten.

Die Funktion f des Problems (1) besitze eine stetige Intervallerweiterung [f] auf $T \times V_m([c]) \times V_n(I(R))$. Weiterhin sollen Konstanten $l_{ij} \geq 0$, i,j = 1,2,...,n existieren, so daß für beliebige $w,z \in R^n$ die Ungleichung

$$|f_i(t,c,w)-f_i(t,c,z)| \leq \sum_{j=1}^{n} l_{ij}|w_j-z_j|,\ i = 1,2,\ldots,n, \qquad (3)$$

für jedes c ∈ [c] und t ∈ T erfüllt ist. Durch die Größen l_{ij} wird die LIPSCHITZ-Matrix $L = (l_{ij})$ definiert und Q sei eine n×n Matrix mit den Elementen:

$$q_{ij} = \begin{cases} 0 & \text{für } i \neq j \\ l_{ij} & \text{für } i = j. \end{cases}$$

Basisschritt: Zunächst wird eine für das Verfahren zulässige Schrittweite h > 0 so gewählt, daß $0 < h \leq d(T)$ und

$$h\|L + 2(L-Q)\| < 1 \qquad (4)$$

gilt. Weiterhin werden folgende Hilfsgrößen definiert:

$$\overline{\alpha} := \max_{c \in [c]} f(\underline{t},c,\overline{a}) \quad , \quad \underline{\alpha} := \min_{c \in [c]} f(\underline{t},c,\underline{a}) \qquad (5)$$

$$\overline{u}(t) := \overline{\alpha}(t-\underline{t})+\overline{a} \quad , \quad \underline{u}(t) := \underline{\alpha}(t-\underline{t})+\underline{a} \qquad (6)$$

$$B_1 := I-h(2L-Q) \quad , \quad B_2 := -h(L-Q)$$

mit

$$\underline{a}_i < \underline{x}_i(\underline{t}) \leq \overline{x}_i(\underline{t}) < \overline{a}_i\ , \quad i = 1,2,\ldots,n.$$

Aus den n×n Matrizen B_1 und B_2 wird die 2n×2n Blockmatrix

$$B := \begin{pmatrix} B_1 & B_2 \\ B_2 & B_1 \end{pmatrix} \tag{7}$$

gebildet. Der Vektor $\beta = (\beta_1,\beta_2,\ldots,\beta_{2n})^T$ sei Lösung des linearen Gleichungssystems

$$B\beta = b \tag{8}$$

mit

$$b = \begin{pmatrix} (-\bar{\alpha} + \max\limits_{\substack{t\in T_1\\ c\in[c]}} f(t,c,\bar{u}) + (L-Q) \max\limits_{t\in T_1} (\bar{u}(t)-\underline{u}(t)) \\ (\underline{\alpha} - \min\limits_{\substack{t\in T_1\\ c\in[c]}} f(t,c,\underline{u}) + (L-Q) \max\limits_{t\in T_1} (\bar{u}(t)-\underline{u}(t))) \end{pmatrix}$$

und

$$T_1 := [\underline{t},\underline{t}+h].$$

Mit Hilfe des Vektors

$$\beta = \begin{pmatrix} \bar{\beta} \\ \underline{\beta} \end{pmatrix}$$

$\bar{\beta},\underline{\beta} \in R^n$ werden die folgenden Funktionen definiert

$$\bar{s}^1(t) := (\bar{\alpha}+\bar{\beta})(t-\underline{t})+\bar{a}$$

$$\underline{s}_1(t) := (\underline{\alpha}-\underline{\beta})(t-\underline{t})+\underline{a}$$

und es gilt der folgende

Satz 1: Die Funktionen $\bar{s}^1$ und $\underline{s}^1$ sind in $\tilde{T}_1 = [\underline{t},\underline{t}+h)$ Ober- und Unterschrankenfunktionen der Lösungsmenge $\{\hat{x}\}$ des Problems

Problems (1), d.h. es gilt für jedes $x \in \{\hat{x}\}$ die Beziehung $\underline{s}^1(t) \le x(t) \le \bar{s}^1(t)$, $t \in \tilde{T}_1$.

Zum Beweis von Satz 1 werden die folgenden Lemmata benötigt.

Lemma 1: Die Matrix B ist inverspositiv, also insbesondere regulär.

Beweis: Mit der Blockmatrix

$$\tilde{B} = \begin{pmatrix} 2L-Q & L-Q \\ L-Q & 2L-Q \end{pmatrix}$$

läßt sich B wie folgt darstellen,

$$B = I-h\tilde{B},$$

wobei $h\|\tilde{B}\|_\infty < 1$ wegen (4) gilt. Weiterhin ist $h\tilde{B} \ge O$ und $\rho(h\tilde{B}) < 1$. Mit dem Hilfssatz aus der Matrizentheorie:

Falls $M \ge O$ eine n×n Matrix ist, dann sind die beiden folgenden Aussagen äquivalent

(1) $\rho(M) < 1$

(2) I-M ist regulär und $(I-M)^{-1} \ge O$,

folgt jetzt

$$B = I-h\tilde{B}$$

ist regulär und inverspositiv.

Lemma 2: Sei β Lösung von (8), dann gilt $\beta \ge O$.

Beweis: Mit (5) gelten die Abschätzungen für i = 1,2,...,n

$$b_i = -\bar{\alpha}_i + \max_{\substack{t\in T_1\\ c\in[c]}} f_i(t,c,\bar{u}(t)) + \sum_{j\neq i} l_{ij}u_j$$

$$\geq -\bar{\alpha}_i + \max_{c \in [c]} f_i(\underline{t},c,\bar{u}(\underline{t})) + \sum_{j \neq i} l_{ij} u_j$$

$$= \sum_{j \neq i} l_{ij} u_j \geq 0$$

$$b_{n+i} = \underline{\alpha}_i - \min_{\substack{c \in [c] \\ t \in T_1}} f_i(t,c,\underline{u}(t)) + \sum_{j \neq i} l_{ij} u_j$$

$$\geq \underline{\alpha}_i - \min_{c \in [c]} f_i(\underline{t},c,\underline{u}(\underline{t})) + \sum_{j \neq i} l_{ij} u_j$$

$$= \sum_{j \neq i} l_{ij} u_j \geq 0 .$$

Mit $b \geq 0$ und $B^{-1} \geq 0$ folgt $\beta \geq 0$.

Lemma 3: Die Funktionen $\bar{u},\underline{u},\bar{s}^1,\underline{s}^1$ erfüllen für $t \in T_1$ die Ungleichungen

(a) $\underline{s}^1(t) \leq \bar{u}(t)$

(b) $\underline{u}(t) \leq \bar{s}^1(t)$

(c) $\underline{s}^1(t) \leq \bar{s}^1(t)$.

Beweis: Nach Lemma 2 gilt

$$\bar{u}(t) = \bar{\alpha}(t-\underline{t}) + \bar{a} \leq (\bar{\alpha}+\bar{\beta})(t-\underline{t}) + \bar{a} = \bar{s}^1(t) \qquad (9)$$

$$\underline{u}(t) = \underline{\alpha}(t-\underline{t}) + \underline{a} \geq (\underline{\alpha}-\underline{\beta})(t-\underline{t}) + \underline{a} = \underline{s}^1(t).$$

Es wird die Ungleichung (a) gezeigt. Es gilt

$$\bar{u}_i(t) - \underline{s}_i^1(t) = (\bar{\alpha}_i - \underline{\alpha}_i + \underline{\beta}_i)(t-\underline{t}) + \bar{a}_i - \underline{a}_i, \quad i = 1,2,\ldots,n. \qquad (10)$$

Weiter verfolgt werden muß nur der Fall $(\bar{\alpha}_i - \underline{\alpha}_i + \underline{\beta}_i) < 0$, da für $(\bar{\alpha}_i - \underline{\alpha}_i + \underline{\beta}_i) \geq 0$ die Ungleichung (a) offensichtlich erfüllt ist.

Aus dem Gleichungssystem (8) ergibt sich unter Berücksichtigung von Lemma 2 und $\max_{t \in T_1} (\bar{u}_j(t) - \underline{u}_j(t)) \geq \bar{a}_j - \underline{a}_j$, $j = 1,2,\ldots,n$ die Abschätzung

$$\underline{\beta}_i = h\, l_{ii}\underline{\beta}_i + h \sum_{j \neq i} (2\underline{\beta}_j + \bar{\beta}_j) + \underline{\alpha}_i - \min_{\substack{c \in [c] \\ t \in T_1}} f_i(t,c,\underline{u}(t))$$

$$+ \sum_{j \neq i} l_{ij} (\bar{u}_j(t) - \underline{u}_j(t)$$

$$\geq \underline{\alpha}_i - \min_{c \in [c]} f_i(\underline{t},c,\underline{u}(t)) + \sum_{j \neq i} l_{ij} (\bar{a}_j - \underline{a}_j)$$

$$= \sum_{j \neq i} l_{ij} (\bar{a}_j - \underline{a}_j) \qquad i = 1,2,\ldots,n.$$

Nach Einsetzen in (10) folgt, da $t-\underline{t} \leq h$,

$$\bar{u}_i(t) - \underline{s}^1(t) \geq (\bar{\alpha}_i - \underline{\alpha}_i + \underline{\beta}_i)h + \bar{a}_i - \underline{a}_i$$

$$= \Big(\max_{c \in [c]} f_i(\underline{t},c,\bar{a}) - \min_{c \in [c]} f_i(\underline{t},c,\underline{a}) + \underline{\beta}_i\Big)h + \bar{a}_i - \underline{a}_i$$

$$\geq (f_i(\underline{t},\tilde{c},\bar{a}) - f_i(\underline{t},\tilde{c},\underline{a}) + \underline{\beta}_i)h + \bar{a}_i - \underline{a}_i$$

$$\geq \Big(-\sum_{j} l_{ij} (\bar{a}_j - \underline{a}_j) + \sum_{j \neq i} l_{ij} (\bar{a}_j - \underline{a}_j)\Big)h + \bar{a}_i - \underline{a}_i$$

$$= (1 - h\, l_{ii})(\bar{a}_i - \underline{a}_i) \geq 0$$

$$i = 1,2,\ldots,n$$

mit $\tilde{c} \in [c]$ beliebig. Damit ist die Ungleichung (a) gezeigt. Der Beweis der Ungleichung (b) verläuft analog und die Ungleichung (c) folgt aus (a), (b) und (9).

Beweis des Satzes 1: Es wird gezeigt, daß $\bar{s}^1$ eine Oberschrankenfunktion für $\{\hat{x}\}$ auf $\tilde{T}_1$ ist. Ausgehend vom Gleichungssystem (8) gilt für ein beliebiges aber festes $c \in [c]$ die Ungleichung

$$(1-h\, l_{ii})\bar{\beta}_i - 2h \sum_{j\neq i} l_{ij}\bar{\beta}_j - h \sum_{j\neq i} l_{ij}\underline{\beta}_j$$

$$\geq -\bar{\alpha}_i + f_i(\underline{t},c,\bar{u}(t)) + \sum_{j\neq i} l_{ij}(\bar{u}_j(t)-\underline{u}_j(t)),\quad i = 1,2,\ldots,n.$$

Nach Umordnen der einzelnen Terme und unter Benutzung von $h > t-\underline{t}$ hat die Ungleichung die Gestalt

$$\bar{\alpha}_i+\bar{\beta}_i - \sum_{j\neq i} ((\bar{\beta}_j+\underline{\beta}_j)(t-\underline{t})+\bar{u}_j(t)-\underline{u}_j(t)) - \sum_j l_{ij}\bar{\beta}_j(t-\underline{t})$$

$$- f_i(t,c,\bar{u}(t)) > 0.$$

Da

$$\bar{s}_i^{1\prime}(t) = \bar{\alpha}_i + \bar{\beta}_i$$

und

$$\bar{s}_i^1(t)-\underline{s}_i^1(t) = (\bar{\alpha}_i+\bar{\beta}_i-\underline{\alpha}_i+\underline{\beta}_i)(t-\underline{t})+\bar{a}_i-a_i$$

lautet die Ungleichung nun

$$\bar{s}_i^{1\prime}(t) - \sum_{j\neq i} (\bar{s}_j^1(t)-\underline{s}_j^1(t)) - \sum_j l_{ij}(\bar{s}_j^1(t)-\bar{u}_j(t))$$

$$- f_i(t,c,\bar{u}(t)) > 0. \qquad (11)$$

Aus Lemma 3 folgt für festes $t \in \tilde{T}_1$ und für alle $w \in R^n$ mit $\underline{s}^1(t) \leq w \leq \bar{s}^1(t)$ die Abschätzung

$$\bar{s}^1(t)-\underline{s}^1(t) \geq |\bar{s}^1(t)-w|.$$

Weiterhin gilt aufgrund der LIPSCHITZ-Bedingung (3) und der Aussagen des Lemmas 3:

$$f_i(t,c,\bar{s}^1(t))-f_i(t,c,\bar{u}(t)) \leq \sum_j l_{ij}(\bar{s}_j^1(t)-\bar{u}_j(t)) \qquad (12)$$

und

$$f_i(t,c,w)-f_i(t,c,\bar{s}^1(t)) \leq \sum_{j\neq i} l_{ij}(\bar{s}_j^1(t)-w_j) \qquad (13)$$

mit $w_i = \bar{s}_i^1(t)$.

Nach Anwendung von (12) und (13) kann die linke Seite von Ungleichung (11) abgeschätzt werden

$$0 < \bar{s}_i^{1\prime}(t) - \sum_{j\neq i} l_{ij}(\bar{s}_j^1(t)-\underline{s}_j^1(t)) - \sum_j l_{ij}(\bar{s}_j^1(t)-\bar{u}_j^1(t))$$

$$- f_i(t,c,\bar{u}(t))$$

$$\leq \bar{s}_i^{1\prime}(t) - \sum_{j\neq i} l_{ij}(\bar{s}_j^1(t)-\underline{s}_j^1(t))-f_i(t,c,\bar{s}^1(t))$$

$$\leq \bar{s}_i^{1\prime}(t) - \sum_{j\neq i} l_{ij}(\bar{s}_j^1(t)-w_j)-f_i(t,c,\bar{s}^1(t))$$

$$\leq \bar{s}_i^{1\prime}(t)-f_i(t,c,w_1,w_2,\ldots,\bar{s}_i^1,\ldots,w_n),$$

$$i = 1,2,\ldots,n.$$

Unter Verwendung des bekannten Lemmas von MÜLLER [2] folgt somit, daß $\bar{s}^1$ Oberschrankenfunktion von $\{\hat{x}\}$ ist, und damit $\bar{s}^1(t) \geq x(t)$, $x \in \{\hat{x}\}$ gilt. Der Beweis, daß $\underline{s}^1(t)$ Unterschrankenfunktion von $\{\hat{x}\}$ ist, verläuft analog.

Da die im Basisschritt konstruierte Anfangseinschließung $[s^1]$ im allgemeinen sehr grob bezüglich ihres Durchmessers ist, ist eine iterative Verbesserung angezeigt. Diese wird durch eine mehrfache

Ausführung des Verbesserungsschrittes erreicht, wobei aufgrund der erweiterten Aufgabenstellung ($d([c]) > 0$) gegenüber [7] und besonders durch das Fehlen der Voraussetzung der Quasimonotonie von f keine Konvergenz der berechneten Folgen von Ober- und Unterschrankenfunktionen gegen die Intervallhülle der Lösungsmenge $\{\hat{x}\}$ erwartet werden kann.

Verbesserungsschritt: Für $k = 1,2,\ldots$ werden die Funktionen

$$\begin{aligned} \overline{s}^{k+1}(t) &:= \overline{s}^{k}(t) - \overline{z}^{k+1}(t) \\ \underline{s}^{k+1}(t) &:= \underline{s}^{k}(t) + \underline{z}^{k+1}(t) \end{aligned} \tag{14}$$

auf $T_2 = [\underline{t}, \underline{t}+h_2]$, $0 < h_2 < h$ gebildet, die Funktionen $\overline{z}^{k+1}$, $\underline{z}^{k+1}$ werden sogleich erklärt. Es werden die Intervallpolynome

$$[p^{k+1}(t)] = \overline{s}'^{k}(t) - [f(t,[c],\overline{s}^{k}(t))] - (L-Q)(\overline{s}^{k}(t) - \underline{s}^{k}(t))$$

$$[r^{k+1}(t)] = [f(t,[c],\underline{s}^{k}(t))] - \underline{s}'^{k}(t) - (L-Q)(\overline{s}^{k}(t) - \underline{s}^{k}(t)),$$

$$k = 1,2,\ldots$$

bestimmt und mit ihrer Hilfe die Funktionen $\overline{z}^{k+1}$, $\underline{z}^{k+1}$ wie folgt definiert:

$$\overline{z}_i^{k+1}(t) := \begin{cases} \int_{\underline{t}}^{t} (1-l_{ii}(\tau-t))\underline{p}_i(\tau)d\tau, & \text{falls } \min_{t\in T_2} \underline{p}_i^{k}(t) \geq 0 \\ 0, & \text{sonst} \end{cases}$$

$$\underline{z}_i^{k+1}(t) := \begin{cases} \int_{\underline{t}}^{t} (1-l_{ii}(\tau-t))\underline{r}_i(\tau)d\tau, & \text{falls } \min_{t\in T_2} \underline{r}_i^{k}(t) \geq 0 \\ 0, & \text{sonst} \end{cases}$$

$$k = 1,2,\ldots .$$

Bezüglich der gemäß (14) und (15) gebildeten Funktionen $\overline{s}^{k+1}$, $\underline{s}^{k+1}$, $k = 1,2,\ldots$ gilt unter der Voraussetzung $(L-Q)\overline{z}^{k+1} > 0$, $(L-Q)\underline{z}^{k+1} > 0$ der folgende

Satz 2: Die Funktionen $\overline{s}^{k+1}$, $\underline{s}^{k+1}$, $k = 1,2,\ldots$ sind Ober- und Unterschrankenfunktionen der Lösungsmenge $\{\hat{x}\}$ auf T_2 und es gilt die Beziehung

$$\underline{s}^{k}(t) \leq \underline{s}^{k+1}(t) \leq x(t) \leq \overline{s}^{k+1}(t) \leq \overline{s}^{k}(t)$$

für jedes $x \in \{\hat{x}\}$, $t \in T_2$ und $k = 1,2,\ldots$.

Bemerkungen:

1. Die Aufgabenklasse wurde eingeschränkt (f_i-Polynom in t und x, $i = 1,2,\ldots,n$), um eine rechentechnische Realisierung zu erleichtern. Im Verlaufe des Verfahrens müssen stets nur Polynome behandelt werden. Das Differenzieren, Integrieren, Addieren und Multiplizieren von Polynomen läßt sich ohne größeren Aufwand rechentechnisch realisieren.
2. Die Schrittweite h hängt nur von der LIPSCHITZ-Matrix ab und braucht nur einmal bestimmt zu werden.
3. Das Verfahren bricht ab, wenn kein $\underline{p}_i(t)$ oder $r_i(t) \geq 0$ $i = 1,2,\ldots,n$ auf T_2 gefunden werden kann oder eine gewünschte Genauigkeit erreicht wurde.
4. Im Zuge der Rechnungen entstehen stetige Ober- und Unterschrankenfunktionen, die nur in höchstens endlich vielen Punkten nicht differenzierbar sind.

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