Robust Methodology for Characterizing System Response to Damage: Approach Based on Partial Order

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Introduction

- *Problem:* describe the response of engineering complex systems to various damage mechanisms.
- *Traditional approach:*
  - use number-valued utilities to describe possible results,
  - use probabilities to describe frequencies.
- *Assumption:* an expert can always make a definite preference (total order).
- *In reality:* preferences are partially ordered.
- *Tank example:* hitting an engine vs. hitting a gun.
- *Objective:* extend decision theory to partial orders.
- *Important particular case:* uncertainty description (S. Markov et al.).

Traditional Utility Theory: In Brief

- *Alternatives:* $\mathcal{A} = \{a_1, \ldots, a_n\}$.
- *Lottery:* $p_1 \cdot a_1 + \ldots + p_n \cdot a_n$, where

$$p_i \geq 0 \text{ and } \sum_{i=1}^{n} p_i = 1.$$

- *Complex lotteries:* when $\ell, \ldots, \ell' \in L$, we can define

$$p \cdot \ell + \ldots + p' \cdot \ell'.$$

- *Preference:* preorder $\preceq$ s.t. when $0 < p < 1$:

$$\ell \preceq \ell' \leftrightarrow p \cdot \ell + (1 - p) \cdot \ell'' \preceq p \cdot \ell' + (1 - p) \cdot \ell''$$

- *Utility:* $u : L \to R$ s.t. $\ell \preceq \ell' \leftrightarrow u(\ell) \leq u(\ell')$, and

$$u(p \cdot \ell + \ldots + p' \cdot \ell') = p \cdot u(\ell) + \ldots + p' \cdot u(\ell').$$

- *Main result:* total ordered preferences $\preceq$ are described by utility functions u.
- *Uniqueness:* if u and u' describe the same $\preceq$, then $u'(\ell) = T(u(\ell))$ for some $T(z) = k \cdot z + m$.

First Auxiliary Notion: Affine Space

- *Affine space:* $\approx$ vector space with no fixed 0.
- *Difference* in more precise terms:
  - a *linear space* is $\langle V, +, \cdot \rangle$; we can define

$$\lambda_1 \cdot v_1 + \ldots + \lambda_n \cdot v_n$$

    where $\lambda_i \in R$ and $v_i \in V$;
  - in *affine space*, we can only define $\sum \lambda_i \cdot v_i$ when $\sum \lambda_i = 1$.
- *Relationship:*
  - *Affine→vector:* if V is affine, we pick any $v_0 \in V$ and make a vector space with $v_0 = 0$:

$$v + v' \stackrel{\text{def}}{=} 1 \cdot v + 1 \cdot v' - 1 \cdot v_0; \quad \lambda \cdot v \stackrel{\text{def}}{=} \lambda \cdot v + (1 - \lambda) \cdot v_0.$$

  - *Vector→affine:* any hyperplane H in a linear space is an affine space.

Second Auxiliary Notion: Ordered Space

- A vector space V with a strict order $<$ is an *ordered vector space* if for every $v, v', v'' \in V$, and for every real number $\lambda > 0$, we have:
  - if $v < v'$, then $v + v'' < v' + v''$;
  - if $v < v'$, then $\lambda \cdot v < \lambda \cdot v'$.
- Since $<$ does not change under shift, it, in effect, defines an ordering on the affine space.
- A *vector utility function* is $u : L \to V$ s.t.

$$\ell \preceq \ell' \leftrightarrow u(\ell) \le u(\ell'), \quad \text{and}$$

$$u(p \cdot \ell + \ldots + p' \cdot \ell') = p \cdot u(\ell) + \ldots + p' \cdot u(\ell').$$

- *Isomorphism* $T : V \to V'$ preserves:
  - affine structure: $T(\sum \lambda_i \cdot v_i) = \sum \lambda_i \cdot T(v_i)$;
  - order: $v < v' \leftrightarrow T(v) < T(v')$.

Main Result:
Consistency and Existence

- *Notations:*
  - let $\mathcal{A}$ be a set, and
  - let L be the set of all lotteries over $\mathcal{A}$.
- *Consistency:*
  - for every convexity-preserving function $u : L \to V$ from L to an ordered affine space,
  - the relation $u(\ell) \leq u(\ell')$ is a preference relation.
- *Existence:*
  - for every preference relation $\preceq$,
  - there exists a vector utility function which describes this preference.

Main Result: Uniqueness

- *In brief:* the utility function is determined uniquely modulo an isomorphism.
- *First part:*
  - If $u : L \to V$ and $u' : L \to V'$ describe the same preference $\preceq$,
  - then there exists an isomorphism $T : A(u(L)) \to A(u'(L))$ (where $A(S)$ is an affine hull),
  - such that for every lottery ℓ, $u'(\ell) = T(u(\ell))$.
- Vice versa:
  - if a vector utility function $u : L \to V$ describes a preference relation,
  - and $T : A(u(L)) \to V'$ is an isomorphism,
  - then $u'(\ell) \stackrel{\text{def}}{=} T(u(\ell))$ is also a vector utility function, and it describes the same preference relation.

Example

- *Example:* tank.
- *Description:* it is natural to describe damage as a vector-valued utility (u_1, u_2), where:
  - u_1 describes the tank's shooting abilities, and
  - u_2 the tank's moving abilities.
- *Towards realistic description:* we also need to take into consideration:
  - communication capabilities u_3,
  - possibility of damage repair u_4, etc.
- *Resulting description:* a higher-dimensional utility vector $(u_1, u_2, u_3, u_4, \ldots)$.

How to Describe Degrees of Belief for Partially Ordered Preferences?

- *Problem:* describe degree of belief ("subjective probability") $ps(E)$ in a statement E.
- *Traditional approach:* pick a_0 and a_1 with utilities 0 and 1, and define $ps(E) \stackrel{\text{def}}{=} u(E|a_1|a_0)$, where

$$(E|a_1|a_0) \stackrel{\text{def}}{=} \text{“if } E \text{ then } a_1 \text{ else } a_0\text{”}$$

- *Motivation:* if E is random w/probability p, then

$$ps(E) = u(E|a_1|a_0) = p \cdot u(a_1) + (1-p) \cdot u(a_0) = p.$$

- *Interpretation:* We have

$$u(E|\ell|\ell') = ps(E) \cdot u(\ell) + (1 - ps(E)) \cdot u(\ell'),$$

hence

$$u(E|\ell|\ell') - u(\ell') = ps(E) \cdot (u(\ell) - u(\ell')).$$

So, $ps(E)$ is a *linear operator.*

Conditional Lotteries

- *Definition:* $\sum p_i \cdot \ell_i + \sum q_k \cdot (E|\ell'_k|\ell''_k)$,

  where $\sum p_i + \sum q_k = 1$, and ℓ_i, ℓ'_k, and ℓ''_k are lotteries.

- *Preference relation* on the set $L(E)$ of all conditional lotteries satisfies additional properties:

  1. if $\ell \sim \ell'$, then $(E|\ell|\ell'') \sim (E|\ell'|\ell'')$;
  2. if $\ell' \sim \ell'''$, then $(E|\ell|\ell') \sim (E|\ell|\ell'')$;
  3. $(E|\ell|\ell) \sim \ell$;
  4. $(E|p \cdot \ell + (1-p) \cdot \ell'|\ell'') \sim$

     $p \cdot (E|\ell|\ell'') + (1-p) \cdot (E|\ell'|\ell'')$;
  5. $(E|\ell|p \cdot \ell' + (1-p) \cdot \ell'') \sim$

     $p \cdot (E|\ell|\ell') + (1-p) \cdot (E|\ell|\ell'')$;
  6. $(E|p \cdot \ell + (1-p) \cdot \ell''|p \cdot \ell' + (1-p) \cdot \ell'') \sim$

     $p \cdot (E|\ell|\ell') + (1-p) \cdot \ell''$;
  7. if $\ell \preceq \ell'$, then $\ell \preceq (E|\ell|\ell') \preceq \ell'$.

Degrees of Belief: First Result

- *Definitions:*
  - A linear operator $T : V \to V$ is *non-negative* (denoted $T \geq \mathbf{0}$) iff $x > 0 \to Tx \geq 0$.
  - T is called a *probability operator* if both T and $\mathbf{1} - T$ are non-negative.
- *First result:*
  - Let $u : L \to V$ be a vector utility function and
  - let $T : V \to V$ be a strict probability operator.
  - Then,

$$u^*\left(\sum_i p_i \cdot \ell_i + \sum_k q_k \cdot (E|\ell'_k|\ell''_k)\right) \stackrel{\text{def}}{=}$$

$$\sum_i p_i \cdot u(\ell_i) + \sum_k q_k \cdot u^*(E|\ell'_k|\ell''_k),$$

with $u^*(E|\ell|\ell') \stackrel{\text{def}}{=} Tu(\ell) + (\mathbf{1} - T)u(\ell')$, is a vector utility function which describes a preference relation on $L(E)$.

Degrees of Belief: Second Result

- Let $\preceq$ be a preference relation on $L(E)$.
- Let $u : L(E) \to V$ be a vector utility function which describes this preference.
- Then, there exists a probability operator

$$T : A(u(L)) \to V$$

for which

$$u(E|\ell|\ell') = Tu(\ell) + (\mathbf{1} - T)u(\ell')$$

for all ℓ and ℓ', and

$$u\left(\sum_i p_i \cdot \ell_i + \sum_k q_k \cdot (E|\ell'_k|\ell''_k)\right) =$$

$$\sum_i p_i \cdot u(\ell_i) + \sum_k q_k \cdot u(E|\ell'_k|\ell''_k).$$

Degrees of Belief: Third Result

- *Reminder:* a degree of belief is described by an operator, i.e., by a matrix.
- *General case:* in general, we need n^2 components to describe an $n \times n$ matrix.
- *Theorem:* the set of all probability operators is at most n-dimensional.
- *Proof:*
  - $\preceq$ is described by a convex cone $P \stackrel{\text{def}}{=} \{v \mid v \geq 0\}$;
  - P is a convex hull of (extreme) generators;
  - let generators $e_1, \ldots, e_n$ form a base for V;
  - T is uniquely determined by values $T(e_i)$;
  - $0 \leq T(e_i) \leq e_i$ hence $T(e_i)$ belongs to the same generator, i.e., $T(e_i) = \lambda_i \cdot e_i$;
  - so, to describe T, it is enough to know n values λ_i.

Degrees of Belief: Final Results

- *Definitions:*
  - *Cartesian product* $V_1 \times V_2$ is the set of all pairs (v_1, v_2) with $v_1 \in V_1$ and $v_2 \in V_2$ for which
  $$(v_1, v_2) \geq 0 \text{ if and only if } v_1 \geq 0 \text{ and } v_2 \geq 0.$$
  - *Lattice order* when in some coordinate system, $(x_1, \ldots, x_n) \geq 0$ iff $x_1 \geq 0$, and $x_2 \geq 0$, ..., and $x_n \geq 0$.
  - $P(V)$ is the set of all probability operators on V.
- *Result* 4*:* $\dim(P(V)) > 1$ iff
$$V = V_1 \times V_2 \text{ for non-degenerate } V_1 \text{ and } V_2.$$
- *Result* 5*:* $\dim(P(V)) = n$ iff V is a lattice order.
- *Conclusion:* for most ordered vector spaces, we need $< n$ parameters.

Proof of Result 4

- If $V = V_1 \times V_2$, then $(v_1, v_2) \to (\lambda_1 \cdot v_1, \lambda_2 \cdot v_2)$ is a probability operator; thus $\dim(P(V)) \ge 2$.
- Let $\dim(P(V)) > 1$; each $T \in P(V)$ is $T(e_i) = \lambda_i \cdot e_i$; so, for some $T \in P(V)$, $\lambda_i \ne \lambda_j$.
- Thus, $V = V_1 \times \ldots \times V_m$, where V_i corr. to diff. λ_i.
- On V_i, we define $v_i \ge 0 \leftrightarrow (0, \ldots, 0, v_i, 0, \ldots, 0) \ge 0$.
- If $v_1 \ge 0$, ..., $v_n \ge 0$, then

  $(v_1, \ldots, v_m) = (v_1, 0, \ldots, 0) + \ldots + (0, \ldots, 0, v_m) \ge 0.$
- Vice versa, if $v = (v_1, \ldots, v_m) \in P$ (i.e., $v \ge 0$), then v is a convex combination of extreme generators.
- Each generator e is an eigenvector of T thus, $\exists i\ e \in V_i$.
- Grouping $e \in V_i$, we get $v = v'_1 + \ldots + v'_m$, $v'_i \ge 0$.
- Due to uniqueness, $v'_i = v_i$ and $v_i \ge 0$.

Proof of Result 5

Lattice order $\rightarrow \dim(P(V)) = n$:

- For a lattice order, for every n values $\lambda_1, \ldots, \lambda_n \in [0, 1]$, the mapping $(x_1, \ldots, x_n) \rightarrow (\lambda_1 \cdot x_1, \ldots, \lambda_n \cdot x_n)$ is a probability operator.
- So, $\dim(P(V)) \geq n$; we know that $\dim(P(V)) \leq n$, hence $\dim(P(V)) = n$.

$\dim(P(V)) = n \rightarrow$ lattice order:

- Vice versa, the only case when we have an n-dimensional set of probability operators is when we have n different eigenspaces.
- All eingespaces have thus to be 1-dimensional.
- In this case, V is a Cartesian order of n real lines, i.e., a lattice order.

Conclusions

- Describing possible damage is important.
- Traditional probability-based approach assumes that preference is a total order.
- In real life, an expert may not be able to always compare two different alternatives.
- We describe decision making under partial order.
- The "utility" is now an element of a (partially) ordered vector space.
- The "probability" is now a matrix.
- At first glance, the necessity to use multi-dimensional "probabilities" leads to an increase in computational complexity.
- In reality, however, for most partial orders, the corresponding "probabilities" are actually 1-dimensional.

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