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Inner estimation of the united solution set of interval linear algebraic system

LUDMILA KUPRIYANOVA

It is shown that an algebraic interval solution of interval linear algebraic systems with matrix composed of "reverse" interval elements of the input matrix is a maximum inner estimation for the united solution set in the sense of inclusion.

Оценка снизу объединенного множества решений интервальных линейных алгебраических систем

Л. КУПРИЯНОВА

Показано, что алгебраическое интервальное решение интервальной линейной алгебраической системы, матрица которой составлена из «инвертированных» интервалов — элементов исходной матрицы, является максимальной оценкой снизу для объединенного множества решений в смысле включения.

Introduction

We will denote intervals and interval vectors by Latin letters $a, b, c, \dots$ (with or without index), interval matrices by capital Latin letters $A, B, C, \dots$ Real vectors and matrices will be denoted by Latin letters with lower dots $\underset{\cdot}{a}, \underset{\cdot}{b}, \underset{\cdot}{c}, \dots, \underset{\cdot}{A}, \underset{\cdot}{B}, \underset{\cdot}{C}, \dots$ Endpoints of intervals and of interval vectors will be denoted by $\underline{a}$, $\overline{a}$ (lower and upper endpoints, respectively), real numbers—by Greek letters $\alpha, \beta, \gamma, \dots$, and also by small Latin letters with lower dots or with bars.

Let us consider a real system of linear equations

$$\underset{\cdot}{A}\underset{\cdot}{x} = \underset{\cdot}{b}$$

where $\underset{\cdot}{a}_{ij}$, $\underset{\cdot}{b}_i$ vary in intervals, that is, $\underset{\cdot}{a}_{ij} \in [\underline{a}_{ij}, \overline{a}_{ij}]$ and $\underset{\cdot}{b}_i \in [\underline{b}_i, \overline{b}_i]$. Also assume that for all $\underset{\cdot}{a}_{ij}$ the matrix $\underset{\cdot}{A}$ is nonsingular.

It is common knowledge that the set of all solutions to real systems obtained by considering all possible values of coefficients from the given intervals (the united solution set) is not an interval vector, and may have a complicated structure. There are many papers devoted to computing outer estimations for this set (see, e.g., [7, 9]).

In this paper a way of finding an interval vector that is an inner estimation for the united solution set is presented.

For this purpose an extended interval arithmetic of Kaucher [2] is used whose objects are intervals $x = [\underline{x}, \overline{x}]$, the requirement $\underline{x} \leq \overline{x}$ being unnecessary. The intervals with $\underline{x} \leq \overline{x}$ (proper intervals) are interpreted by us as the sets of points ξ on the real axis satisfying $\underline{x} \leq \xi \leq \overline{x}$. The intervals with $\underline{x} > \overline{x}$ (improper intervals) are used by us for intermediate computations as auxiliary objects. For example, suppose we wish to find an algebraic interval solution of the interval equation

$$[-1, 1] + x = [0, 3]$$

that is, that interval x such that substitution of x into this equation renders it an identity. The interval $[1, -1]$, additive inverse for $[-1, 1]$, is an improper interval and is not interpreted as set of real numbers, but it aids finding a solution

$$x = [0, 3] + [1, -1] = [1, 2].$$

Substituting x into the equation, we obtain the identity

$$[-1, 1] + [1, 2] = [0, 3].$$

The interval $[1, 2]$ is proper and has a real meaning.

Thus, the extended interval set, compared to the classical interval set, has better properties, namely, the existence of additive inverses (for all intervals) and the existence of multiplicative inverses (for intervals not containing zero). Besides, the arithmetic introduced in a suitable way allows us to obtain the result by effecting equivalent transformations. So, given an initial problem dealing with proper intervals (with sets of real numbers) we use the objects of the extended interval set in the solution process. If the obtained solution is a proper interval then the problem is considered to be solved.

In Section 1, we cite the relevant elements of theory concerning the extended set of intervals and also a theorem needed for a proof that the resulting interval vector is contained in the united solution set.

In Section 2, properties of the algebraic interval solution for the system with the dual matrix and a fundamental theorem on maximum inner estimation for the united solution set are proposed.

In Section 3, we present a way for finding an algebraic interval solution, proposed originally by Zyuzin [12, 13] for the systems with coefficients from $I(R)$. It is shown that the algorithm also keeps its properties in more general cases and may have more general-purpose application than just the inner estimation problem.

In Section 4, some examples illustrating the properties of algebraic interval solutions for the system with dual matrix are presented.

1. Some properties of an extended interval set

An extended interval arithmetic was proposed by Kaucher [2]. We will use it in the form given by Gardenes, Trepat [1].

Let $I(R) := \{x = [\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in R,\ \underline{x} \leq \overline{x}\}$ be a set of intervals with positive widths and

$$I^*(R) := \{x = [\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in R\}$$

be a set that contains intervals both with positive width (proper) and with negative width (improper). The detailed presentation of properties of the set $I^*(R)$ can be found in [1, 2]. We cite only those that are necessary here.

Definition 1.1. *The magnitude* $w(x) = \overline{x} - \underline{x}$ *will be called the width of the interval* $x = [\underline{x}, \overline{x}] \in I^*(R)$.

Definition 1.2. *We shall refer to an interval* $x \in I^*(R)$ *as a proper interval if* $\underline{x} \leq \overline{x}$ *and as an improper interval if* $\underline{x} > \overline{x}$.

Definition 1.3. *(of unary operations on* $I^*(R)$*)*

$$\begin{aligned} \mathrm{dual}(x) &:= [\overline{x}, \underline{x}]; \\ \mathrm{opp}(x) &:= [-\underline{x}, -\overline{x}]; \\ \mathrm{pr}(x) &:= \begin{cases} [\underline{x}, \overline{x}] & \text{if } x \text{ is proper}, \\ [\overline{x}, \underline{x}] & \text{if } x \text{ is improper}. \end{cases} \end{aligned}$$

Definition 1.4. *(of inclusion on* $I^*(R)$*)*

$$(x \subseteq y) \Leftrightarrow (\underline{y} \leq \underline{x} \;\&\; \overline{x} \leq \overline{y}).$$

If one of inequalities is strong, i.e. $(x \subseteq y)$ & $(x \neq y)$*, then we shall write "*$x \subset y$*".*

Definition 1.5.

$$\begin{aligned} (x \leq y) &\Leftrightarrow \Big(\big(\forall \underset{\cdot}{x} \in \mathrm{pr}(x)\big)\big(\forall \underset{\cdot}{y} \in \mathrm{pr}(y)\big)\; \underset{\cdot}{x} \geq \underset{\cdot}{y} \Big) \\ (x < y) &\Leftrightarrow \Big(\big(\forall \underset{\cdot}{x} \in \mathrm{pr}(x)\big)\big(\forall \underset{\cdot}{y} \in \mathrm{pr}(y)\big)\; \underset{\cdot}{x} > \underset{\cdot}{y} \Big). \end{aligned}$$

The real numbers $\underset{\cdot}{x} \in R$ are identified with intervals such that $\underset{\cdot}{x} = [\underset{\cdot}{x}, \underset{\cdot}{x}] \in I^*(R)$. Then the record "$x \geq 0$", given $0 = [0, 0]$, means "$\big(\forall \underset{\cdot}{x} \in \mathrm{pr}(x)\big)\; \underset{\cdot}{x} \geq 0$". Similarly "$0 \subseteq x$" means "$\underline{x} \leq 0 \leq \overline{x}$" and "$x \subseteq 0$" means "$\overline{x} \leq 0 \leq \underline{x}$".

To define arithmetic operations over the elements of $I^*(R)$, let us introduce a characteristic of interval direction which will be denoted by Ω.

Definition 1.6. *Let* $a_i \in I^*(R)$*. Define the lattice operations*

$$\begin{aligned} \bigvee_i a_i &:= sup_{\subseteq}\, a_i &= [\inf_{\leq} \underline{a}_i, \sup_{\leq} \overline{a}_i]; \\ \bigwedge_i a_i &:= inf_{\subseteq}\, a_i &= [\sup_{\leq} \underline{a}_i, \inf_{\leq} \overline{a}_i] \end{aligned}$$

on a bounded set of a_i *where* i *can run through a finite set or a countable set or continuum.*

Definition 1.7. *Characteristic of interval direction*

$$\Omega^x := \begin{cases} \bigvee & \text{if } x \text{ is proper}; \\ \bigwedge & \text{if } x \text{ is improper}. \end{cases}$$

It follows from the definition that $x = \underset{\underset{\cdot}{x} \in \mathrm{pr}(x)}{\Omega^x} \underset{\cdot}{x}$.

Definition 1.8. *(of arithmetic operations)*

$$a * b := \underset{\underset{\cdot}{a} \in \mathrm{pr}(a)}{\Omega^a}\; \underset{\underset{\cdot}{b} \in \mathrm{pr}(b)}{\Omega^b}\; \underset{\cdot}{a} * \underset{\cdot}{b}, \quad * \in \{+, -, \cdot, /\}.$$

According to this definition all operations can be described as operations with endpoints of intervals in the following way.

Addition

$$a + b = [\underline{a} + \underline{b}, \overline{a} + \overline{b}];$$

$$a - b = [\underline{a} - \overline{b}, \overline{a} - \underline{b}].$$

Multiplication

1) if $a \geq 0$ & $b \geq 0$ then $ab = [\underline{a}\underline{b}, \overline{a}\overline{b}]$;

2) if $a \geq 0$ & $b \leq 0$ then $ab = [\overline{a}\underline{b}, \underline{a}\overline{b}]$;

3) if $a \geq 0$ & $b \supseteq 0$ then $ab = [\overline{a}\underline{b}, \overline{a}\overline{b}]$;

4) if $a \geq 0$ & $b \subseteq 0$ then $ab = [\underline{a}\underline{b}, \underline{a}\overline{b}]$;

5) if $a \leq 0$ & $b \leq 0$ then $ab = [\overline{a}\overline{b}, \underline{a}\underline{b}]$;

6) if $a \leq 0$ & $b \supseteq 0$ then $ab = [\underline{a}\overline{b}, \underline{a}\underline{b}]$;

7) if $a \leq 0$ & $b \subseteq 0$ then $ab = [\overline{a}\overline{b}, \overline{a}\underline{b}]$;

8) if $a \supseteq 0$ & $b \supseteq 0$ then $ab = [min\{\underline{a}\overline{b}, \overline{a}\underline{b}\}, max\{\underline{a}\underline{b}, \overline{a}\overline{b}\}]$;

9) if $a \subseteq 0$ & $b \subseteq 0$ then $ab = [max\{\underline{a}\underline{b}, \overline{a}\overline{b}\}, min\{\underline{a}\overline{b}, \overline{a}\underline{b}\}]$;

10) if $a \subseteq 0$ & $b \supseteq 0$ then $ab = 0$.

Division (for $0 \notin \mathrm{pr}(b)$)

$$1/b = [1/\overline{b}, 1/\underline{b}];$$

$$a/b = a \cdot (1/b).$$

Algebraic division (solution of equation $bx = a$ for $0 \notin \mathrm{pr}(b)$)

$$1/.b := [1/\underline{b}, 1/\overline{b}];$$

$$a/.b = a \cdot (1/b).$$

A detailed presentation of properties of the operations introduced above can be found in [1]. Here the subdistributivity property, some generalized concept of a united interval extension, and also the inclusion monotonicity property for the introduced operations will be necessary.

Theorem 1.1. *(on subdistributivity) Let $a, b, c \in I^*(R)$.*

If a is a proper interval then $a(b+c) \subseteq ab + ac$;
if a is an improper interval then $a(b+c) \supseteq ab + ac$.

Definition 1.9. *If $f(x, y)$ is a real function,*

$x = (x_1, x_2, \ldots, x_p)^T$ *is a proper interval vector,*
$y = (y_1, y_2, \ldots, y_q)^T$ *is an improper interval vector,*

we define "interval extensions" of f as

$$f^*(x,y) := \bigvee_{\underset{\cdot}{x}\in x} \bigwedge_{\underset{\cdot}{y}\in \mathrm{pr}(y)} f(\underset{\cdot}{x}\underset{\cdot}{y}),$$

$$f^{**}(x,y) := \bigwedge_{\underset{\cdot}{y}\in \mathrm{pr}(y)} \bigvee_{\underset{\cdot}{x}\in x} f(\underset{\cdot}{x}\underset{\cdot}{y}).$$

Remark. In general these two functions can be distinct. However, we shall consider a function f, where each interval occurs only once, and for this function, $f^* = f^{**}$, and both f^* and f^{**} coincide with the natural interval extension obtained when intervals replace the real variables and interval arithmetic operations replace corresponding real operations.

The following theorem from [1] will be needed later.

Theorem 1.2. *If $f^*(x,y)$ is proper then*

$$(\forall \underset{\cdot}{x} \in x)\big(\exists \underset{\cdot}{y} \in \mathrm{pr}(y)\big) \quad f(\underset{\cdot}{x},\underset{\cdot}{y}) \in f^*(x,y).$$

It is not difficult to show that inclusion monotonicity of interval operations is valid for all elements of $I^*(R)$.

Theorem 1.3. *Interval operations introduced by Definition 1.7 are monotone with respect to inclusion, i.e. for $a,b,x,y \in I^*(R)$*

$$(a \subseteq b \;\&\; x \subseteq y) \Rightarrow (a * x \subseteq b * y), \quad \text{where } * \in \{+,-,\cdot,/\}.$$

2. Algebraic interval solution for the system with dual matrix

Let us consider the real linear algebraic system

$$\underset{\cdot}{A}\underset{\cdot}{x} = \underset{\cdot}{b} \tag{2.1}$$

the coefficients of which vary in intervals such that $\underset{\cdot}{A} \in A$, $\underset{\cdot}{b} \in b$, where A is a proper interval $n \times n$-matrix, b is a proper interval n-vector, and $(\forall \underset{\cdot}{A} \in A)\, det(\underset{\cdot}{A}) \neq 0$.

Definition 2.1. *The united solution set (or a union of all solutions of the real system (2.1)) is that set of real vectors*

$$x^* := \{\underset{\cdot}{x} \mid \underset{\cdot}{A}\underset{\cdot}{x} = \underset{\cdot}{b}, \underset{\cdot}{A} \in A, \underset{\cdot}{b} \in b\}. \tag{2.2}$$

It follows from the definition that

$$(\forall \underset{\cdot}{x} \in x^*)\ (\exists \underset{\cdot}{A} \in A)\ (\exists \underset{\cdot}{b} \in b) \quad \underset{\cdot}{A}\underset{\cdot}{x} = \underset{\cdot}{b}. \tag{2.3}$$

In general, x^* is not an interval vector.

Problem. To find a maximal inner estimation for x^*, i.e. a maximal interval vector z satisfying the property (2.3)

$$(\forall \underset{\cdot}{x} \in z)\ (\exists \underset{\cdot}{A} \in A)\ (\exists \underset{\cdot}{b} \in b) \quad \underset{\cdot}{A}\underset{\cdot}{x} = \underset{\cdot}{b}.$$

To solve this problem, let us consider an interval system

$$Ax = b. \tag{2.4}$$

Definition 2.2. *An interval vector x^a satisfying the system (2.4), i.e. an interval vector which when substituted into (2.4) makes the equations of this system the true equalities, will be called an algebraic interval solution of this system.*

In general, algebraic interval solutions are not unique; additional investigation is necessary to determinate conditions for uniqueness. However, Shary [8] has given conditions that imply uniqueness for proper matrices. Uniqueness is not relevant to the properties studied here.

Definition 2.3. *A set of real vectors [3, 5, 6]*

$$x_{\subseteq} := \{\underset{\cdot}{x} \mid A\underset{\cdot}{x} \subseteq b\}$$

will be called a tolerable solution set.

It follows from the definition that

$$(\forall \underset{\cdot}{x} \in x_{\subseteq})\ (\forall \underset{\cdot}{A} \in A)\ (\exists \underset{\cdot}{b} \in b) \quad \underset{\cdot}{A}\underset{\cdot}{x} = \underset{\cdot}{b} \tag{2.5}$$

$x_{\subseteq} \subseteq x^*$, and also that $x_{\subseteq}$ is not an interval vector in general. However, if the system (2.5) has the algebraic interval solution x^a, all components of which are proper, then x^a is an inner estimation of the tolerable solution set, $x^a \subseteq x_{\subseteq}$.

Indeed, if $Ax^a = b$ with $a_{ij}, x_j, b_j \in I(R)$ $(i = \overline{1,n},\ j = \overline{1,n})$, then

$$b_i = a_{i1}x_1^a + a_{i2}x_2^a + \cdots + a_{in}x_n^a.$$

It follows from the definitions of interval arithmetic operations for proper intervals that the above equality implies

$$(\forall \underset{\cdot}{x}_j \in x_j^a, j = \overline{1,n})\ (\forall \underset{\cdot}{a}_{ij} \in a_{ij}, j = \overline{1,n}) \quad \underset{\cdot}{a}_{i1}\underset{\cdot}{x}_1 + \cdots + \underset{\cdot}{a}_{in}\underset{\cdot}{x}_n \subseteq b_i$$

and for the whole system

$$(\forall \underset{\cdot}{x} \in x^a)\ (\forall \underset{\cdot}{A} \in A) \quad \underset{\cdot}{A}\underset{\cdot}{x} \subseteq b$$

that is equivalent to

$$(\forall \underset{\cdot}{x} \in x^a)\ (\forall \underset{\cdot}{A} \in A)\ (\exists \underset{\cdot}{b} \in b) \quad \underset{\cdot}{A}\underset{\cdot}{x} = \underset{\cdot}{b}.$$

Details of the tolerable solution set and on methods for finding its inner estimation appear in papers of Shary (e.g. [5, 6]).

An algebraic interval solution for a system with dual matrix

$$\mathrm{dual}(A)x = b \tag{2.6}$$

where $\mathrm{dual}(A)_{ij} := \mathrm{dual}(a_{ij}) = \mathrm{dual}[\underline{a}_{ij}, \overline{a}_{ij}] = [\overline{a}_{ij}, \underline{a}_{ij}]$, will be denoted by x^d.

Details concerning algebraic interval solutions are possible to see in papers of Zyuzin [12, 13] and Zakharov [9–11]. We wish to show that the algebraic interval solution of the system (2.6) x^d has some useful properties, formulated in the following theorems.

Theorem 2.1. *Any algebraic interval solution of the system (2.6) with proper components $\left(x_i^d \in I(R)\right)$ is included in the united solution set x^* to $Ax = b$, that is,*

$$x^d \subseteq x^*.$$

Proof. Let us consider the i-th row of the system (2.1)

$$(\underset{\cdot}{A}\underset{\cdot}{x})_i = \underset{\cdot}{a}_{i1} \cdot \underset{\cdot}{x}_1 + \underset{\cdot}{a}_{i2} \cdot \underset{\cdot}{x}_2 + \cdots + \underset{\cdot}{a}_{in} \cdot \underset{\cdot}{x}_n = \underset{\cdot}{b}_i.$$

Let $\underset{\sim}{a}_i := (\underset{\sim}{a}_{i1}, \underset{\sim}{a}_{i2}, \ldots, \underset{\sim}{a}_{in})$ be a row of the matrix $\underset{\sim}{A}$, $\underset{\sim}{x} = (\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n)^T$, and $f_i(\underset{\sim}{a}_i, \underset{\sim}{x}) := \underset{\sim}{a}_i \cdot \underset{\sim}{x}$. Given a real function f_i, intervals

$$\mathrm{dual}(a_i) = \Big(\mathrm{dual}(a_{i1}), \ldots, \mathrm{dual}(a_{in})\Big)$$

and

$$x^d = (x_1^d, \ldots, x_n^d)$$

where dual(a) is an improper vector and x^d is proper, we shall consider the interval extension

$$f_i^* := \bigvee_{\underset{\sim}{x} \in x^d} \bigwedge_{\underset{\sim}{a}_i \in a_i} f_i(\underset{\sim}{a}_i, \underset{\sim}{x}) = \mathrm{dual}(a_{i1})x_1^d + \mathrm{dual}(a_{i2})x_2^d + \cdots + \mathrm{dual}(a_{in})x_n^d = b_i \tag{2.7}$$

where f_i^* is a row of the system (2.6), and, for f_i, the functions f_i^* and f_i^{**} correspond to the natural interval extension. Then according to Theorem 1.2 we have

$$(\forall \underset{\sim}{x} \in x^d)\ \Big(\exists \underset{\sim}{a}_{ik} \in a_{ik}\ (k = \overline{1,n})\Big)\quad \underset{\sim}{a}_{i1} \cdot \underset{\sim}{x}_1 + \underset{\sim}{a}_{i2} \cdot \underset{\sim}{x}_2 + \cdots + \underset{\sim}{a}_{in} \cdot \underset{\sim}{x}_n \in b_i$$

or

$$(\forall \underset{\sim}{x} \in x^d)\ \Big(\exists \underset{\sim}{a}_{ik} \in a_{ik}\ (k = \overline{1,n})\Big)\ (\exists \underset{\sim}{b}_i \in b_i)\quad \underset{\sim}{a}_{i1} \cdot \underset{\sim}{x}_1 + \cdots + \underset{\sim}{a}_{in} \cdot \underset{\sim}{x}_n = \underset{\sim}{b}_i$$

and for the whole system

$$(\forall \underset{\sim}{x} \in x^d)\ (\exists \underset{\sim}{A} \in A)\ (\exists \underset{\sim}{b} \in b)\quad \underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{b}.$$

Hence, every real vector $\underset{\sim}{x}$ from x^d is a solution of some real system, i.e. we have proven $x^d \subseteq x^*$. □

To prove Theorem 2.3 below, an inner estimation property of x^d, the properties in the following lemmas are necessary.

Lemma 2.1. *Let $a, b \in I(R)$, $d \in I^*(R)$, and $a \cap b \neq \emptyset$.*

$$\textit{If } a + d \in I(R) \textit{ and } \underline{d} > w(a) + w(b) \textit{ then } (a + d) \cap b = \emptyset.$$

(Let two proper intervals a and b intersect. Then the proper intervals $a + d$ and b do not intersect if the lower endpoint of d is greater than the sum of widths of intervals a and b).

Lemma 2.2. *Let $a, b \in I(R)$, $d \in I^*(R)$, and $a \cap b \neq \emptyset$.*

$$\textit{If } a + d \in I(R) \textit{ and } \overline{d} < -w(a) - w(b) \textit{ then } (a + d) \cap b = \emptyset.$$

(Let two proper intervals a and b intersect. Then the proper intervals $a + d$ and b do not intersect if the upper endpoint of d is less than the sum of widths of intervals a and b taken with opposite sign).

The assertions of Lemmas 2.1 and 2.2 can be easily checked.

Lemma 2.3. *Let $a, x \in I(R)$ be proper intervals. Let the real numbers $x', x'' \in x$ take values of endpoints of the interval x depending in the following way on the signs of a and x:*

1) if $a > 0$ then $x' = \underline{x}$, $x'' = \overline{x}$;

2) if $a < 0$ then $x' = \overline{x}$, $x'' = \underline{x}$;

3) if $a \supseteq 0$ & $x \geq 0$ then $x' = x'' = \underline{x}$;

4) if $a \supseteq 0$ & $x \leq 0$ *then* $x' = x'' = \overline{x}$.

Then the equality $ax'' = ax' + d$ *is valid, where* $d \in I^*(R)$ *and* $\underline{d} = w(ax') + w\big(\mathrm{dual}(a) \cdot x\big)$.

Lemma 2.4. *Let* $a, x \in I(R)$ *be proper intervals. Let the real numbers* $x', x'' \in x$ *take values of endpoints of the interval* x *depending on the signs of* a *and* x *in the following way:*

1) if $a > 0$ *then* $x' = \overline{x}$, $x'' = \underline{x}$;

2) if $a < 0$ *then* $x' = \underline{x}$, $x'' = \overline{x}$;

3) if $a \supseteq 0$ & $x \geq 0$ *then* $x' = x'' = \underline{x}$;

4) if $a \supseteq 0$ & $x \leq 0$ *then* $x' = x'' = \overline{x}$.

Then the equality $ax'' = ax' + d$ *is valid, where* $d \in I^*(R)$ *and* $\overline{d} = -w(ax') - w\big(\mathrm{dual}(a) \cdot x\big)$.

The validity of these Lemmas can be easily checked using the rules of multiplication 1)–10) from Section 1.

Theorem 2.2 (*Fundamental theorem on inner estimation*).

An algebraic interval solution x^d *of the system with dual matrix (2.6) , with* $x_j^d \in I(R)$ $(j = \overline{1,n})$, *is a maximum inner estimation for the united solution set of the system (2.1) in the sense of inclusion. That is*

$$(x^d \subseteq x^*) \ \& \ \Big((\forall \tilde{x} \supset x^d)\ \tilde{x} \not\subseteq x^*\Big).$$

Proof. By Theorem 2.1 the interval vector x^d satisfying the system $\mathrm{dual}(A) \cdot x = b$ is included in the united solution set x^* of the system $\underset{\cdot}{A}\underset{\cdot}{x} = \underset{\cdot}{b}$, where $\underset{\cdot}{A} \in A$, $\underset{\cdot}{b} \in b$, i.e. it is an inner estimation for x^*. Now we wish to prove that this estimation is maximum in the sense of inclusion, that is $(\forall \tilde{x} \supset x^d)\ \tilde{x} \not\subseteq x^*$.

In other words, we must show that any vector wider than x^d includes points which do not belong to x^* and, hence, are solutions for no real system (2.1).

Let $\tilde{x} = (x_1^d, \ldots, x_{j-1}^d, [\underline{x}_j^d, \overline{x}_j^d + \varepsilon], x_{j+1}^d, \ldots, x_n^d)$ be an interval vector whose j-th coordinate is greater by $\varepsilon > 0$ than x^d. It is necessary to show that $\tilde{x}$ overflows the boundaries of x^*, i.e.

$$(\exists \xi \in \tilde{x})\ (\forall \underset{\cdot}{A} \in A)\ (\forall \underset{\cdot}{b} \in b) \quad \underset{\cdot}{A}\xi \neq \underset{\cdot}{b}$$

or at least for some row

$$(\exists \xi \in \tilde{x})\ (\forall \underset{\cdot}{A} \in A)\ (\forall \underset{\cdot}{b} \in b) \quad (\underset{\cdot}{A}\xi)_i \neq \underset{\cdot}{b}_i.$$

Let us take a row such that $0 \notin \mathrm{int}(a_{ij})$. This is always possible to do. Indeed, if for all i, $a_{ij} \ni 0$ were true, then A would contain a real matrix $\underset{\cdot}{A}$ with a zero column, but the assumption is $\forall \underset{\cdot}{A} \in A\, det(\underset{\cdot}{A}) \neq 0$.

I. Assuming $a_{ij} > 0$, we choose $\xi \in \tilde{x}$ so that

$$\begin{array}{llcl} \text{for } k \neq j & \xi_k & = & \begin{cases} \overline{x}_k^d, & \text{if } (a_{ik} > 0) \vee \Big((a_{ik} \supseteq 0) \ \& \ (x_k^d < 0)\Big) \\ \underline{x}_k^d, & \text{if } (a_{ik} < 0) \vee \Big((a_{ik} \supseteq 0) \ \& \ (x_k^d > 0)\Big) \\ 0, & \text{if } (a_{ik} \supseteq 0) \ \& \ (x_k^d \supseteq 0) \end{cases} \\ \text{and} & \xi_j & = & \overline{x}_j^d + \varepsilon. \end{array}$$

Now for ξ we wish to show that $\left((\forall \underset{\cdot}{A} \in A)(\forall \underset{\cdot}{b} \in b)\ (A\xi)_i \neq b_i\right) \Leftrightarrow \left((\forall \underset{\cdot}{A} \in A)\ (A\xi)_i \notin b_i\right) \Leftrightarrow \left((A\xi)_i \cap b_i = \emptyset\right)$, given that A is proper. Thus, it is necessary to verify that $(A\xi)_i$ and b_i are disjoint. Let us choose $\underset{\cdot}{x} \in x^d$ so

$$\text{for } k \neq j \qquad \underset{\cdot}{x}_k = \begin{cases} \underline{x}_k^d, & \text{if } (a_{ik} > 0) \vee \left((a_{ik} \supseteq 0)\ \&\ (x_k^d > 0)\right) \\ \overline{x}_k^d, & \text{if } (a_{ik} < 0) \vee \left((a_{ik} \supseteq 0)\ \&\ (x_k^d < 0)\right) \\ 0, & \text{if } (a_{ik} \supseteq 0)\ \&\ (x_k^d \supseteq 0) \end{cases}$$
$$\text{and} \qquad \underset{\cdot}{x}_j = \underline{x}_j^d.$$

This choice of ξ and $\underset{\cdot}{x}$ ensures the conditions of Lemma 2.3 for all $k \neq j$, if we take a_{ik}, x_k, $\underset{\cdot}{x}_k$, and ξ_k as a, x, x', and x'' respectively.

Then Lemma 2.3 implies

$$a_{ik} \cdot \xi_k = a_{ik} \cdot \underset{\cdot}{x}_k + d_k, \quad \text{where } d_k \in I^*(R) \text{ and } \underline{d}_k = w(a_{ik} \cdot \underset{\cdot}{x}) + w\left(\operatorname{dual}(a_{ik}) \cdot x_k\right). \tag{2.8}$$

The case not treated by the Lemma, $a_{ik} \supseteq 0$, $x_k^d \supseteq 0$, is trivial: taking $\xi_k = 0$, $\underset{\cdot}{x}_k = 0$ results in $a_{ik} \cdot \xi_k = a_{ik} \cdot \underset{\cdot}{x}_k = 0$ and $d_k = [0,0]$,

$$w(a_{ik} \cdot \underset{\cdot}{x}_k) + w\left(\operatorname{dual}(a_{ik}) \cdot x_k^d\right) = w([0,0]) + w([0,0]) = 0 = \underline{d}_k.$$

For $k = j$

$$a_{ij} \cdot \xi_j = a_{ij} \cdot (\overline{x}_j^d + \varepsilon).$$

Since $a_{ij} > 0$ by assumption, the following cases are possible:

a) $x_j^d > 0 \Rightarrow \overline{x}_j^d \geq \underline{x}_j^d > 0$

$$\begin{aligned} a_{ij} \cdot \xi_j &= a_{ij} \cdot (\overline{x}_j^d + \varepsilon) = [\underline{a}_{ij} \cdot (\overline{x}_j^d + \varepsilon), \overline{a}_{ij} \cdot (\overline{x}_j^d + \varepsilon)] \\ &= [\underline{a}_{ij} \cdot \underline{x}_j^d + \underline{a}_{ij} \cdot (\overline{x}_j^d - \underline{x}_j^d + \varepsilon), \overline{a}_{ij} \cdot \underline{x}_j^d + \overline{a}_{ij} \cdot (\overline{x}_j^d - \underline{x}_j^d + \varepsilon)] \\ &= a_{ij} \cdot \underline{x}_j^d + [\underline{a}_{ij} \cdot (\overline{x}_j^d - \underline{x}_j^d + \varepsilon), \overline{a}_{ij} \cdot (\overline{x}_j^d - \underline{x}_j^d + \varepsilon)] \\ &= a_{ij} \cdot \underset{\cdot}{x}_j + d_j \end{aligned}$$

where $\underline{d}_j = \underline{a}_{ij} \cdot (\overline{x}_j^d - \underline{x}_j^d) + \underline{a}_{ij} \cdot \varepsilon$.

$$\begin{aligned} w(a_{ij}\underset{\cdot}{x}_j) + w\left(\operatorname{dual}(a_{ij})x_j\right) &= w([\underline{a}_{ij}\underline{x}_j^d, \overline{a}_{ij}\underline{x}_j^d]) + w([\overline{a}_{ij}, \underline{a}_{ij}] \cdot [\underline{x}_j^d, \overline{x}_j^d]) \\ &= \overline{a}_{ij}\underline{x}_j^d - \underline{a}_{ij}\underline{x}_j^d + w([\overline{a}_{ij}\underline{x}_j^d, \underline{a}_{ij}\overline{x}_j^d]) \\ &= \overline{a}_{ij}\underline{x}_j^d - \underline{a}_{ij}\underline{x}_j^d + \underline{a}_{ij}\overline{x}_j^d - \overline{a}_{ij}\underline{x}_j^d \\ &= \underline{a}_{ij}(\overline{x}_j^d - \underline{x}_j^d) < \underline{d}_j. \end{aligned}$$

b) $x_j^d < 0 \Rightarrow \underline{x}_j^d \leq \overline{x}_j^d < 0$.

Taking ε sufficiently small so that $\overline{x}^d + \varepsilon < 0$ (if for a small increment one can find a point not belonging x^* then the more so for a larger one), one can write

$$\begin{aligned} a_{ij} \cdot \xi_j &= a_{ij} \cdot (\overline{x}_j^d + \varepsilon) = [\overline{a}_{ij} \cdot (\overline{x}_j^d + \varepsilon), \underline{a}_{ij} \cdot (\overline{x}_j^d + \varepsilon)] \\ &= [\overline{a}_{ij} \cdot \underline{x}_j^d + \overline{a}_{ij} \cdot (\overline{x}_j^d - \underline{x}_j^d + \varepsilon), \underline{a}_{ij} \cdot \underline{x}_j^d + \underline{a}_{ij} \cdot (\overline{x}_j^d - \underline{x}_j^d + \varepsilon)] \\ &= a_{ij} \cdot \underline{x}_j^d + [\overline{a}_{ij} \cdot (\overline{x}_j^d - \underline{x}_j^d + \varepsilon), \underline{a}_{ij} \cdot (\overline{x}_j^d - \underline{x}_j^d + \varepsilon)] \\ &= a_{ij} \cdot \underset{\cdot}{x}_j + d_j \end{aligned}$$

where $\underline{d}_j = \overline{a}_{ij} \cdot (\overline{x}_j^d - \underline{x}_j^d) + \overline{a}_{ij} \cdot \varepsilon$.

$$\begin{aligned} w(a_{ij}x_j) + w\Big(\operatorname{dual}(a_{ij})x_j\Big) &= w([\overline{a}_{ij}\underline{x}_j^d, \underline{a}_{ij}\underline{x}_j^d]) + w([\overline{a}_{ij}, \underline{a}_{ij}] \cdot [\underline{x}_j^d, \overline{x}_j^d]) \\ &= \underline{a}_{ij}\underline{x}_j^d - \overline{a}_{ij}\underline{x}_j^d + w([\underline{a}_{ij}\underline{x}_j^d, \overline{a}_{ij}\overline{x}_j^d]) = \\ &= \underline{a}_{ij}\underline{x}_j^d - \overline{a}_{ij}\underline{x}_j^d + \overline{a}_{ij}\overline{x}_j^d - \underline{a}_{ij}\underline{x}_j^d \\ &= \overline{a}_{ij}(\overline{x}_j^d - \underline{x}_j^d) < \underline{d}_j. \end{aligned}$$

c) $x_j^d \supseteq 0 \Rightarrow \underline{x}_j^d \le 0 \le \overline{x}_j^d$.

$$\begin{aligned} a_{ij} \cdot \xi_j &= a_{ij} \cdot (\overline{x}_j^d + \varepsilon) = [\underline{a}_{ij} \cdot (\overline{x}_j^d + \varepsilon), \overline{a}_{ij} \cdot (\overline{x}_j^d + \varepsilon)] \\ &= [\overline{a}_{ij} \cdot \underline{x}_j^d + (\underline{a}_{ij}\overline{x}_j^d - \overline{a}_{ij}\underline{x}_j^d + \underline{a}_{ij}\varepsilon), \underline{a}_{ij} \cdot \underline{x}_j^d + (\overline{a}_{ij}\overline{x}_j^d - \underline{a}_{ij}\underline{x}_j^d + \overline{a}_{ij}\varepsilon)] \\ &= a_{ij} \cdot x_j + d_j \end{aligned}$$

where $\underline{d}_j = (\underline{a}_{ij}\overline{x}_j^d - \overline{a}_{ij}\underline{x}_j^d) + \underline{a}_{ij} \cdot \varepsilon$.

$$\begin{aligned} w(a_{ij}x_j) + w\Big(\operatorname{dual}(a_{ij})x_j\Big) &= w([\overline{a}_{ij}\underline{x}_j^d, \underline{a}_{ij}\underline{x}_j^d]) + w([\overline{a}_{ij}, \underline{a}_{ij}] \cdot [\underline{x}_j^d, \overline{x}_j^d]) \\ &= \underline{a}_{ij}\underline{x}_j^d - \overline{a}_{ij}\underline{x}_j^d + w([\underline{a}_{ij}\underline{x}_j^d, \underline{a}_{ij}\overline{x}_j^d]) \\ &= \underline{a}_{ij}\underline{x}_j^d - \overline{a}_{ij}\underline{x}_j^d + \underline{a}_{ij}\overline{x}_j^d - \underline{a}_{ij}\underline{x}_j^d \\ &= (\underline{a}_{ij}\overline{x}_j^d - \overline{a}_{ij}\underline{x}_j^d) < \underline{d}_j. \end{aligned}$$

Thus, we see that $\underline{d}_k = w(a_{ik} \cdot x_k) + w\Big(\operatorname{dual}(a_{ik})x_k^d\Big)$ for $k \neq j$, and for $k = j$ $\underline{d}_k > w(a_{ik} \cdot x_k) + w\Big(\operatorname{dual}(a_{ik})x_k^d\Big)$.

Let us denote $d = \sum_{k=1}^n d_k$. Then

$$\begin{aligned} \underline{d} = \sum_{k=1}^n \underline{d}_k &> \sum_{k=1}^n \Big(w(a_{ik} \cdot x_k) + w\Big(\operatorname{dual}(a_{ik})x_k^d\Big)\Big) \\ &= \sum_{k=1}^n w(a_{ik} \cdot x_k) + \sum_{k=1}^n w\Big(\operatorname{dual}(a_{ik})x_k^d\Big) \\ &= w\left(\sum_{k=1}^n a_{ik} \cdot x_k\right) + w\left(\sum_{k=1}^n \operatorname{dual}(a_{ik})x_k^d\right) \\ &= w\Big((Ax)_i\Big) + w\Big(\Big(\operatorname{dual}(A)x_i^d\Big)\Big) \\ &= w\Big((A \cdot x)_i\Big) + w(b_i). \end{aligned} \tag{2.9}$$

Given (2.8) and (2.9), it may be written for whole i-th row

$$(A\xi)_i = \sum_{k=1}^n a_{ik} \cdot \xi_k = \sum_{k=1}^n (a_{ik} \cdot x_k + d_k) = \sum_{k=1}^n a_{ik} \cdot x_k + \sum_{k=1}^n d_k = (Ax)_i + d$$

i.e.

$$(A\xi)_i = (Ax)_i + d \tag{2.10}$$

where $\underline{d} > w\Big((Ax)_i\Big) + w(b_i)$.

For intervals $(Ax)_i$ and b_i, the conditions of Lemma 2.1 are as follows:

$$(Ax)_i, b_i \in I(R), \quad d \in I^*(R) \qquad (Ax)_i \cap b_i \neq \emptyset.$$

(The intervals intersect because $\underline{x} \in x^d \subseteq x^*$ belongs to the united solution set.)

$$(A\underline{x})_i + d = (A\xi)_i \in I(R)$$

(because A is proper and ξ is a real vector) and

$$\underline{d} > w\big((A\underline{x})_i\big) + w(b_i)$$

so Lemma 2.3 implies $\big((Ax)_i + d\big) \cap b_i = \emptyset$, i.e. $(A\xi)_i \cap b_i = \emptyset$, the intervals $(A\xi)_i$ and b_i are disjoint. We see that $(\exists\xi \in \tilde{x})\ (A\xi)_i \cap b_i = \emptyset$, i.e. ξ doesn't belong to the united solution set x^*.
II. For $a_{ij} < 0$ the proof is similar. We choose $\xi \in \tilde{x}$ and $\underline{x} \in x^d$ so that the conditions of Lemma 2.4 would hold

$$\begin{aligned}
\text{for } k \neq j \quad \xi_k &= \begin{cases} \underline{x}^d_k, & \text{if } (a_{ik} > 0) \vee \big((a_{ik} \supseteq 0)\ \&\ (x^d_k < 0)\big) \\ \overline{x}^d_k, & \text{if } (a_{ik} < 0) \vee \big((a_{ik} \supseteq 0)\ \&\ (x^d_k > 0)\big) \\ 0, & \text{if } (a_{ik} \supseteq 0)\ \&\ (x^d_k \supseteq 0) \end{cases} \\
\text{and} \quad \xi_j &= \overline{x}^d_j + \varepsilon, \\
\text{for } k \neq j \quad \underline{x}_k &= \begin{cases} \overline{x}^d_k, & \text{if } (a_{ik} > 0) \vee \big((a_{ik} \supseteq 0)\ \&\ (x^d_k > 0)\big) \\ \underline{x}^d_k, & \text{if } (a_{ik} < 0) \vee \big((a_{ik} \supseteq 0)\ \&\ (x^d_k < 0)\big) \\ 0, & \text{if } (a_{ik} \supseteq 0)\ \&\ (x^d_k \supseteq 0) \end{cases} \\
\text{and} \quad \underline{x}_j &= \underline{x}^d_j.
\end{aligned}$$

Then we will obtain $(A\xi)_i = (A\underline{x})_i + d$, where $\overline{d} < -w\big((A\underline{x})_i\big) - w(b_i)$, and conditions of Lemma 2.2:

$$\begin{aligned}
&(A\underline{x})_i, b_i \in I(R), \quad d \in I^*(R) \\
&(A\underline{x})_i \cap b_i \neq \emptyset \\
&(A\underline{x})_i + d = (A\xi)_i \in I(R) \\
&\text{and} \quad \overline{d} < -w\big((A\underline{x})_i\big) - w(b_i)
\end{aligned}$$

then Lemma 2.2 implies $\big((Ax)_i + d\big) \cap b_i = \emptyset$.

The proof for $\tilde{x} = (x^d_1, \ldots, x^d_{j-1}, [\underline{x}^d_j - \varepsilon, \overline{x}^d_j], x^d_{j+1}, \ldots, x^d_n)$ is similar. □

3. Finding algebraic interval solutions for interval linear algebraic systems

As follows from the foregoing, a maximal inner estimation for the united solution set of an interval linear algebraic system with the proper coefficients is an algebraic interval solution for the system with dual matrix, i.e. for the system with improper matrix and proper right-hand side. Besides, there are tasks that reduce to solving systems with both proper and improper interval coefficients. Therefore, it is desirable to know how to find this solution in the most general case when $a_{ij}, b_i, x^a_j \in I^*(R)$ $(i = \overline{1,n},\ j = \overline{1,n})$.

In [12, 13] an iterative algorithm was proposed for finding an algebraic interval solution x^a provided that the elements of the matrix A , vector b and solution vector x^a are proper intervals. However, the same scheme can be applied to system (2.6) and to systems of general form without any assumptions on "properness" of intervals. We shall prove this here. To construct and to substantiate the algorithm we use the general theory of solution of operator equations [4].

Let us introduce a linear multiplication by real numbers on $I^*(R)$

$$\begin{aligned} \lambda \circ x &:= [\lambda \underline{x}, \lambda \overline{x}]; \\ (-1) \circ x &= [-\underline{x}, -\overline{x}] = \mathrm{opp}(x). \end{aligned}$$

Evidently, $I^*(R)$ with the operations of addition and linear multiplication by real numbers is a linear space. Analogously, the set

$$I^*(R^n) := \{x = (x_1, \ldots, x_n)^T \mid x_k \in I^*(R), k = \overline{1,n}\}$$

with the corresponding componentwise operations is also a linear space. It is easy to see that $I^*(R^n)$ with a norm, for example,

$$\|x\| := \max_k \max\{|\underline{x}_k|, |\overline{x}_k|\}$$

is a complete linear normed space, i.e. a Banach space.

For elements of $I^*(R^n)$ we assign a partial ordering with respect to inclusion $\subseteq$.

Definition 3.1. $x, y \in I^*(R^n)$

$$(x \subseteq y) \Leftrightarrow (y \ominus x \in K)$$

where $y \ominus x := y + (-1) \circ x = y + \mathrm{opp}(x)$, $K := \{x \in I^*(R^n) \mid \underline{x}_k \leq 0 \leq \overline{x}_k, k = \overline{1,n}\}$ *is a cone of Banach space [4].*

(It is obvious that Definition 3.1 is consistent with Definition 1.4 for elements of $I^*(R)$).

Definition 3.2. *Let* $v, w \in I^*(R^n)$. *A set* $<v, w> := \{x \mid v \subseteq x \subseteq w\}$ *will be called a conic segment.*

The following theorem is used for the construction of an iterative method.

Theorem 3.1. *Let* K *be a regular cone,* G *a continuous antitone operator on the conic segment* $<v_0, w_0>$, *and assume that* G *transforms* $<v_0, w_0>$ *to itself. Then the operator* G *has at least one fixed point on* $<v_0, w_0>$, *and the sequences*

$$v_k = Gw_{k-1}, \quad w_k = Gv_k, \quad k = 1, 2, \ldots$$

converge to fixed points v^* *and* w^* *of* G, *and also*

$$v_0 \subseteq v_1 \subseteq \cdots \subseteq v^* \subseteq w^* \subseteq \cdots \subseteq w_1 \subseteq w_0.$$

The proof is analogous to similar theorem on isotone operators proposed in [4].

Now consider the interval system

$$Ax = b \tag{3.1}$$

without any assumption on "properness" of the intervals and only assuming that $\forall \underset{.}{A} \in \mathrm{pr}(A) \det(\underset{.}{A}) \neq 0$. We transform (3.1) to the form

$$x = Bx \tag{3.2}$$

where

$$(Bx)_i := \Big(b_i \ominus \sum_{\substack{k=1\\k\neq i}}^{n} a_{ik}x_k\Big)/.a_{ii}, \quad i = \overline{1,n}.$$

Without loss of generality we can assume $0 \notin a_{ii}$. Indeed, if $\forall A\, det(A) \neq 0$ then by means of a simple transposition of the rows we can always arrange it so that the intervals of the main diagonal do not contain zeroes. Evidently, an algebraic interval solution of (3.2) is simultaneously an algebraic interval solution of (3.1).

Theorem 3.2. *If the operator B of (3.2) transforms the conic segment $<v_0, w_0>$ to itself, then B has at least one fixed point on $<v_0, w_0>$ and the sequences*

$$v_k = Bw_{k-1}, \quad w_k = Bv_k, \quad k = 1, 2\dots$$

converge to fixed points v^* and w^* of B. Also,

$$v_0 \subseteq v_1 \subseteq \cdots \subseteq v^* \subseteq w^* \subseteq \cdots \subseteq w_1 \subseteq w_0.$$

Remark. In all cases, with proper or improper coefficients, each v_k is an inner estimation and each w_k is an outer estimation for the algebraic interval solution. For example, to find an inner estimation to the tolerable solution set to $Ax = b$, with A and b proper, we may take any v_k. Similarly, for an inner estimation to the united solution set to the proper system $Cx = d$, we set $A = \mathrm{dual}(C)$, $b = d$, and take any v_k. In other cases, depending on the original problem, the v_k or w_k may be useful.

Proof. Kaucher [2] has proved the continuity of interval operations in his extended arithmetic. The operator B can be written by interval arithmetic operations as

$$(Bx)_i := \left(b_i + \mathrm{opp}\left(\sum_{k=1}^{n} a_{ik}x_k\right)\right) \cdot (1/.a_{ii}), \quad i = \overline{1,n}.$$

But $0 \notin \mathrm{pr}(a_{ii})$, so $\exists 1/.a_{ii} \in I^*(R)$. From continuity of the corresponding arithmetic operations it follows that B is continuous.

Now we shall prove that the operation $+\mathrm{opp}$ or $\ominus$ is antitone:

$$\text{if } x \subseteq y \text{ then } a \ominus x \supseteq a \ominus y.$$

$$\begin{aligned}
&a \ominus x = [\underline{a} - \underline{x}, \overline{a} - \overline{x}], \quad a \ominus y = [\underline{a} - \underline{y}, \overline{a} - \overline{y}]\\
&(x \subseteq y) \Leftrightarrow (\underline{y} \leq \underline{x} \;\&\; \overline{x} \leq \overline{y}) \Leftrightarrow (-\underline{x} \leq -\underline{y} \;\&\; -\overline{y} \leq -\overline{x}) \Leftrightarrow\\
&(\underline{a} - \underline{x} \leq \underline{a} - \underline{y} \;\&\; \overline{a} - \overline{y} \leq \overline{a} - \overline{x}) \Leftrightarrow (a \ominus x \supseteq a \ominus y).
\end{aligned}$$

By virtue of antitonicity of $a \ominus x$ and of isotonity of interval arithmetic operations 1.8 (see the Theorem 1.3) B is an antitone operator. Thus B is a continuous antitone operator, and Theorem 3.1 is valid for it. □

Choosing initial approximation. If we succeed in finding $v_0 \subseteq x^a = Bx^a$, then because of antitonicity of B: $Bv_0 \supseteq Bx^a$ and hence $v_0 \subseteq Bv_0$. Furthermore, denoting $w_0 = Bv_0$, we have

1) $v_0 \subseteq x^a \subseteq w_0$ there are two-sided initial approximation for x^a;

2) $(v_0 \subseteq w_0) \Rightarrow (Bw_0 \subseteq Bv_0 = w_0)$.

We have

$$v_0 \subseteq Bv_0 \ \& \ Bw_0 \subseteq w_0.$$

Thus if $v_0 \subseteq Bw_0$, then the conic segment $<v_0, w_0>$ is transformed to itself, namely, the left endpoint is transformed to right one and the right endpoint is transformed inside the segment. If B is a contraction operator then $v_0 \subseteq Bw_0$ always. But that is not true in general. Specific rules for good choices of initial approximation have not yet been formulated in general. However, when inner estimation of the united solution set, the main topic of this paper, is desired, the real solution of the system

$$\mathrm{mid}(A) \cdot x = \mathrm{mid}(b)$$

consisting of the midpoints of the interval coefficients of (2.4) can be taken as an initial approximation.

4. Examples

Here, examples of inner estimation of the united solution set by means of computation of an algebraic interval solution are presented.

Example 1 (see Figure 1).

$$A = \begin{pmatrix} [2,4] & [-2,1] \\ [-1,2] & [2,4] \end{pmatrix}, \quad b = \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix}, \quad x^a = \begin{pmatrix} [-1/3, 1/3] \\ [-1/3, 1/3] \end{pmatrix},$$
$$\mathrm{dual}(A) = \begin{pmatrix} [4,2] & [1,-2] \\ [2,-1] & [4,2] \end{pmatrix}, \quad x^d = \begin{pmatrix} [-1,1] \\ [-1,1] \end{pmatrix}.$$

Example 2 (see Figure 2).

$$A = \begin{pmatrix} [2,4] & [-1,1] \\ [-1,1] & [2,4] \end{pmatrix}, \quad b = \begin{pmatrix} [0,2] \\ [0,2] \end{pmatrix}, \quad x^a = \begin{pmatrix} [0.2, 0.4] \\ [0.2, 0.4] \end{pmatrix},$$
$$\mathrm{dual}(A) = \begin{pmatrix} [4,2] & [1,-1] \\ [1,-1] & [4,2] \end{pmatrix}, \quad x^d = \begin{pmatrix} [0,1] \\ [0,1] \end{pmatrix}.$$

Example 3 (see Figure 3).

$$A = \begin{pmatrix} [1,2] & [-2,-1] \\ [0,1] & [3,4] \end{pmatrix}, \quad b = \begin{pmatrix} [1,2] \\ [1,3] \end{pmatrix}, \quad x^a = \begin{pmatrix} [1\frac{11}{12}, 1\frac{1}{6}] \\ [\frac{1}{3}, \frac{11}{24}] \end{pmatrix},$$
$$\mathrm{dual}(A) = \begin{pmatrix} [2,1] & [-1,-2] \\ [1,0] & [4,3] \end{pmatrix}, \quad x^d = \begin{pmatrix} [1,2] \\ [0,1] \end{pmatrix}.$$

Example 4 (see Figure 4).

$$A = \begin{pmatrix} [2,4] & [-1,1] \\ [-1,1] & [2,4] \end{pmatrix}, \quad b = \begin{pmatrix} [-3,3] \\ 0 \end{pmatrix}, \quad x^a = \begin{pmatrix} [-3/4, 3/4] \\ [3/8, -3/8] \end{pmatrix}$$

x^a doesn't exist in the set of proper intervals,

$$\mathrm{dual}(A) = \begin{pmatrix} [4,2] & [1,-1] \\ [1,-1] & [4,2] \end{pmatrix}, \quad x^d = \begin{pmatrix} [-1.5, 1.5] \\ 0 \end{pmatrix}.$$

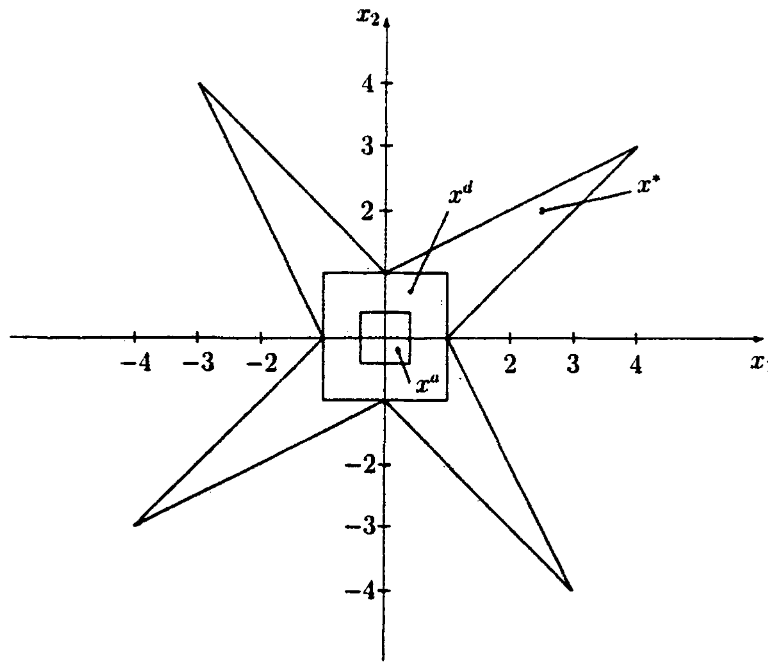

Figure 1.

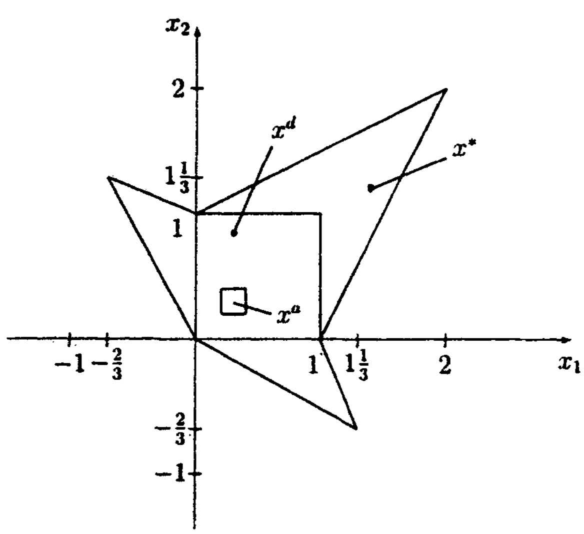

Figure 2.

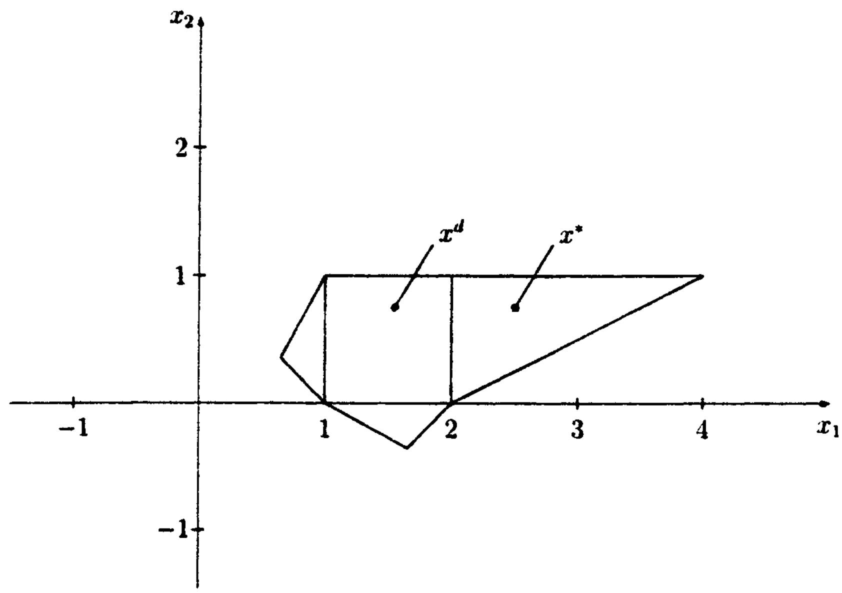

Figure 3.

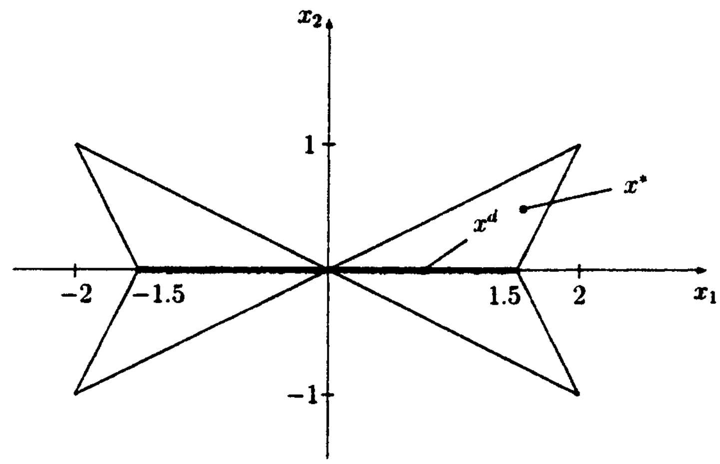

Figure 4.

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Saratov department
of Moscow Commercial University
Saratov, Russia