Reliable Computing 1 (1) (1995), pp. 9–14

Formulas for the width of interval products

HELMUT RATSCHEK* and JON G. ROKNE

Sharp formulas for the width of the product of intervals are derived which are simpler and more effective than the ones previously known. These formulas are useful in applications and they are appropriate tools for estimating the quality of interval evaluations. Proofs of such formulas will, in general, result in a number of different cases involving longwinded calculations. By utilizing certain functionals which are invariants of appropriate interval transformations the calculations are reduced to the ones required for a minimum number of cases.

Формулы для вычисления ширины интервальных произведений

Х. РАЧЕК, ДЖ. ДЖ. РОКН

Представлены точные формулы для вычисления ширины произведения интервалов, являющиеся как более простыми, так и более эффективными, чем использовавшиеся до сих пор. Эти формулы могут быть полезны для приложений; кроме того, они представляют собой хороший инструмент для оценки качества интервального оценивания. Доказательства таких формул, вообще говоря, требуют разбора большого числа случаев и, соответственно, объемных вычислений. Применение функционалов, инвариантных по отношению к определенным интервальным преобразованиям, позволило сократить до минимума число случаев и объем вычислений.

1. Introduction

Let I be the set of real compact intervals. The product in I is defined by

$$AB = \{ab : a \in A, b \in B\} \quad \text{for} \quad A, B \in I$$

or equivalently,

$$[a,b][c,d] = [\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}]$$

cf. [9]. The *width* of an interval $A = [a,b]$ is denoted by $wA = b - a$, the *modulus* by $|A| = \max\{|a|, |b|\}$, the *midpoint* by $mA = (a+b)/2$, the *size* by $sA = (|a| + |b|)/2$, and the (Hausdorff-) *distance* from A to 0 by $< A >= \min\{|a| : a \in A\}$.

It is often necessary to know the values of the above-mentioned functionals when they are applied to an interval product AB. Formulas of varying degree of complexity are known for these formulas (cf. [1, 2, 12, 17]) except for the width $w(AB)$. A surprisingly large number of conditionally valid formulas for $w(AB)$ are known and in part listed below, whereas a universally valid formula has been missing.

*Thanks are due to the National Sciences and Engineering Research Council of Canada for supporting this paper.

Knowledge of $w(AB)$ is required in various kinds of investigations. For example, it is required in the exploitation of the absolute precision concept introduced in [10, 11], in the description of optimal bounds in a general floating point error analysis [12, 17], in proving the existence of higher order approximations of the range of univariate functions [4], in the development of efficient bisection strategies for various problems (cf. [6, 15]; see also [5] for a thorough discussion of bisection strategies), and in recommendations for the evaluation of arithmetic expressions which involve products. Last, but not least, the knowledge of $w(AB)$ is very important in convergence proofs for Newton algorithms (cf. [7]) as well as in numerical validation of solutions of nonlinear systems [3].

In order to hint at the interesting variety of attempts at describing $w(AB)$, we give an excerpt of known formulas keeping as close as possible to the historical development. The first not completely trivial estimation

$$w(AB) \leq |A|wB + |B|wA$$

was proposed in [8]. Later Ris [17] proposed inter alia formulas like

$$w(AB) \leq (sA)wB + (sB)wA.$$

He notes that in this estimation strict inequality holds iff $0 \in \mathring{A}$ and $0 \in \mathring{B}$. (Here, $\mathring{A}$ denotes the interior of A.) Further formulas from [17] are

$$\begin{array}{ll} w(AB) = 2s(AB) = |A|wB & \text{if } 0 \in \mathring{A},\ 0 \notin \mathring{B}, \\ w(AB) = 2s(AB) = \max\{|A|wB, |B|wA\} & \text{if } 0 \in \mathring{A}, 0 \in \mathring{B}, \\ (wA)(wB)/2 \leq w(AB) < (wA)wB & \text{if } 0 \in \mathring{A},\ 0 \in \mathring{B} \end{array}$$

where equality holds on the left hand side iff A and B are symmetric (that is, $A = -A$, $B = -B$).

Alefeld-Herzberger [1, 2] proved inter alia formulas such as

$$\begin{array}{rcll} w(AB) & \geq & \max\{(wA)|B|, |A|wB\}, & \\ w(AB) & = & |B|wA & \text{if } A \text{ is symmetric}, \\ w(A^n) & \leq & n|A|^{n-1}wA, & n = 1, 2, \ldots \end{array}$$

where $A^n = A \times \cdots \times A$ (n times) means the *simple power* in contrast to the *extended power*, $\bar{A}^n = \{a^n : a \in A\}$, and

$$w(A-a)^n \leq 2(wA)^n \quad \text{if } a \in A,\ n = 1, 2, \ldots$$

The last formula could be improved by Rump [18] to

$$w(A-a)^n \leq (wA)^n \quad \text{if } a \in A,\ n = 1, 2, \ldots$$

As far as we know the first complete formula for $w(AB)$ was found by Rall [12]. This formula clearly showed how difficult it was to find a suitable formula. The formula is

$$w(AB) = \max_{i=1,\ldots,6} \alpha_i$$

where

$$\begin{aligned}
\alpha_1 &= |(mA)wB - (wA)(wB)/2|, \\
\alpha_2 &= |(wA)mB - (wA)(wB)/2|, \\
\alpha_3 &= |(wA)mB + (mA)(wB)|, \\
\alpha_4 &= |(wA)mB - (mA)(wB)|, \\
\alpha_5 &= |(wA)mB + (wA)(wB)/2|, \\
\alpha_6 &= |(mA)wB + (wA)(wB)/2|.
\end{aligned}$$

Krawczyk [7] wrote Rall's formula in a more compact form

$$\begin{aligned}
w(AB) = \max \ &\{|mA|wB + (wA)|mB|, \\
&|mA|wB + (wA)(wB)/2, \\
&(wA)|mB| + (wA)(wB)/2\}.
\end{aligned}$$

In Section 2 of this paper sharp formulas for $w(AB)$ are derived. A treatment of such formulas will, in general, result in a number of different cases and longwinded calculations. By utilizing certain functionals which are invariances of the necessary transformations the calculations can be reduced to just one simple case. The most important functional, which characterizes the symmetry behavior of intervals and meets the intrinsic nature of interval products exactly, was introduced in [13] by $\chi : I \to [-1, 1]$ with $\chi[0, 0] = -1$ and if $[a, b] \neq 0$, with

$$\chi[a, b] = \begin{cases} a/b & \text{if } |a| \leq |b| \\ b/a & \text{otherwise.} \end{cases}$$

For example, $\chi A = -1$ means that A is symmetric and $\chi A = 1$ means that A is a nonzero point interval and hence completely unsymmetric. Therefore χ admits the geometric interpretation that

$$A \text{ is } \textit{more symmetric than } B \text{ iff } \chi A \leq \chi B.$$

Beyond that, χ turned out to be an indispensible tool for solving linear interval equations in an algebraic manner [16] or describing the subdistributive behavior of interval arithmetic [14].

2. Formulas for the width of an interval product

Let $A = [a, b]$ and $B = [c, d]$ and $\mathring{A}$ be the interior of an interval A.

Theorem 1. *For intervals A, B with $0 \notin \mathring{A}$, $0 \notin \mathring{B}$ we get*

$$w(AB) = |mA|wB + |mB|wA.$$

Proof. We only need to derive the formula for the case $mA, mB \geq 0$. The remaining cases can be reduced to this case by substituting $A \longmapsto -A$ or $B \longmapsto -B$ or both and observing that the formula of the theorem is invariant w.r.t. the substitution since $w(AB) = w(-AB)$, $|m(-A)| = |mA|$ and $w(-A) = wA$, etc. The assumptions imply a unique explicit expression for the product, $AB = [ac, bd]$. Hence

$$w(AB) = bd - ac = b(d - c) + c(b - a)$$

as well as

$$w(AB) = bd - ac = d(b-a) + a(d-c).$$

Adding the two lines we get

$$\begin{aligned} 2w(AB) &= (c+d)(b-a) + (a+b)(d-c) = 2[(mA)wB + (mB)wA] \\ &= 2[|mA|wB + |mB|wA]. \end{aligned}$$

□

In order to get an expression for $w(AB)$ not containing m or w we apply the obvious formulas

$$|mX| = |X| - (wX)/2 \quad \text{if } X \in I,\ 0 \notin \overset{\circ}{X} \tag{1}$$

$$wX = |X| - \langle X \rangle \quad \text{if } X \in I,\ 0 \notin \overset{\circ}{X} \tag{2}$$

to Theorem 1 with $X = A$ and $X = B$ and gain the following useful formulas.

Corollary 1. *For intervals A, B with $0 \notin \overset{\circ}{A}$, $0 \notin \overset{\circ}{B}$ we get*

$$\text{(i)} \quad w(AB) = |A|wB + |B|wA - (wA)wB, \tag{3}$$

$$\text{(ii)} \quad w(AB) = |A||B| - \langle A \rangle\langle B \rangle. \tag{4}$$

The theorem as well as the corollary shows that when $0 \notin \overset{\circ}{A}\overset{\circ}{B}$ the expressions for $w(AB)$ are symmetric in A and B. The following theorem dealing with the case $0 \in AB$ reveals the surprising feature that $w(AB)$ depends only on the width of the most symmetrical of the two intervals and only on the modulus of the other one.

Theorem 2. *For intervals A, B with $0 \in A$ or $0 \in B$ or both we get*

$$w(AB) = \begin{cases} |A|wB & \text{if } \chi B \le \chi A \quad (5) \\ |B|wA & \text{if } \chi B \ge \chi A. \quad (6) \end{cases}$$

Proof. As in the proof of Theorem 1 we assume $mA, mB \ge 0$ because, in addition to the invariants mentioned in that proof, we have $|A| = |-A|$ and $\chi A = \chi(-A)$. Further we assume $\chi B \le \chi A$. The formula for $\chi B \ge \chi A$ may then be derived by just swapping A and B. Firstly, if $B = 0$, the formula is evident. If $B \ne 0$ then $\chi B = c/d$. Using the homeomorphic properties of the modulus of intervals and the χ-functional as well as the representation of intervals X by $|X|$ and χX, cf. [13], we get

$$\begin{aligned} AB &= |AB|[\chi(AB), 1] \\ &= |A||B|[\chi B, 1] \\ &= |A|\, d\, [c/d, 1] \\ &= |A|[c, d]. \end{aligned}$$

Hence, $w(AB) = |A|wB$. □

Corollary 2. *For intervals A, B we get*

$$w(AB) = \begin{cases} |A|wB + (wA)\langle B \rangle & \text{if } \chi B \le \chi A \\ |B|wA + (wB)\langle A \rangle & \text{if } \chi A \le \chi B. \end{cases} \tag{7}$$

Proof. We restrict ourselves to the case $\chi B \le \chi A$. If $\chi B \ge 0$ (saying that $0 \notin \mathring{B}$), formulas (3) and (2) are identical by setting $X = B$ in (2). If $\chi B \le 0$ (saying that $0 \in B$), then $< B >= 0$ such that formulas (2) and (2) are identical. □

We utilize the binary operation $\ominus$ on $[-1, 1]$, cf. [14], defined by

$$a \ominus b = \begin{cases} ab, & \text{if } a, b \ge 0, \\ \min\{a, b\} & \text{otherwise} \end{cases}$$

and keep in mind that χ is a homomorphism from the algebraic system $\langle I, \cdot \rangle$ to the algebraic system $\langle [-1, 1], \ominus \rangle$, cf. [13].

Theorem 3. *For intervals A, B we get*

$$w(AB) = |A||B|\big(1 - (\chi A) \ominus \chi B\big).$$

Proof. Again $mA, mB \ge 0$ is assumed. Then as in the proof of Theorem 2,

$$\begin{aligned} AB &= |AB|[\chi(AB), 1] \\ &= |A||B|[(\chi A) \ominus \chi B, 1] \end{aligned}$$

and $w(AB) = |A||B|\big(1 - (\chi A) \ominus \chi B\big)$. □

Theorem 3 is of importance if interval vectors or interval products are to be estimated where the previous theorems are not so useful. For example, if $A_i, B_i \in I, i = 1, \dots, n$, then

$$w\left(\sum_{i=1}^{n} A_i B_i\right) = \sum_{i=1}^{n} |A_i||B_i|\big(1 - (\chi A_i) \ominus \chi B_i\big).$$

Corollary 3. *For intervals $A_1, A_2, \dots, A_n$ and A we get*

(i)	$w(A_1 \dots A_n)$	$=$	$\lvert A_1\rvert \cdot \ldots \cdot \lvert A_n\rvert - < A_1 > \cdot \ldots \cdot < A_n >$	*if* $0 \notin \mathring{A}_i, i = 1, \dots, n.$
(ii)	$w(A_1 \dots A_n)$	$=$	$\lvert A_1\rvert \cdot \ldots \cdot \lvert A_{n-1}\rvert wA_n$	*if* $0 \in A_n$ *and* $\chi A_n \le \chi A_i, i = 1, \dots, n,$
(iii)	$w(A^n)$	$=$	$\lvert A^{n-1}\rvert wA$	*if* $0 \in A$

where A^n means the simple power evaluation.

Proof. The modulus as well as the Hausdorff-distance of intervals are homomphisms of $\langle I, \cdot \rangle$ to $\langle R, \cdot \rangle$. Setting $B = A_1 \cdot \ldots \cdot A_{n-1}$ and $A = A_n$ in (4) and in (2) proves (i) and (ii). □

References

[1] Alefeld, G. and Herzberger, J. *Einführung in die Intervallrechnung*. Bibliographisches Institut, Zürich, 1974.

[2] Alefeld, G. and Herzberger, J. *Introduction to interval computations*. Academic Press, New York, 1983.

[3] Alefeld, G., Gienger, A., and Potra, F. *Efficient numerical validation of solutions of nonlinear systems*. SIAM J. Numer. Anal. **31** (1994), pp. 252–260.

[4] **Cornelius, H. and Lohner, R.** *Computing the range of values of real functions with accuracy higher than second order.* Computing **33** (1984), pp. 331–347.

[5] **Kearfott, R. B.** *Abstract generalized bisection and a cost bound.* Math. of Comput. **49** (1987), pp. 187–202.

[6] **Kearfott, R. B.** *Preconditioners for the interval Gauss-Seidel method.* SIAM J. Numer. Anal. **27** (1990), pp. 804–822.

[7] **Krawczyk, R. and Neumaier, A.** *An improved interval Newton operator.* J. Math. Anal. Appl. **118** (1986), pp. 194–207.

[8] **Moore, R. E.** *The automatic analysis and control of error in digital computation based on the use of interval numbers.* In: Rall, L. B. (ed.) "Error in Digital Computation", Vol. 1, Proceedings of an advanced seminar, University of Madison, 1965, pp. 61–130.

[9] **Moore, R. E.** *Interval analysis.* Prentice-Hall, Englewood Cliffs, N.J., 1966.

[10] **Olver, F. W.** *A new approach to error arithmetic.* SIAM J. Numer. Analysis 2 (1982), pp. 368–392.

[11] **Olver, F. W.** *Further developments of rp and ap analysis.* IMA J. Numer. Analysis 2 (1982), pp. 249–274.

[12] **Rall, L. B.** *Representations of intervals and optimal error bounds.* Math. of Computation **41** (1983), pp. 219–227.

[13] **Ratschek, H.** *Die binären Systeme der Intervallmathematik.* Computing **6** (1970), pp. 295–308.

[14] **Ratschek, H.** *Die Subdistributivität in der Intervallmathematik.* Z. Angew. Math. Mech. **51** (1971), pp. 189–192.

[15] **Ratschek, H. and Rokne, J.** *The transistor modeling problem again.* Microelectronics and Reliability **32** (1992), pp. 1725–1740.

[16] **Ratschek, H. and Sauer, W.** *Linear interval equations.* Computing **28** (1982), pp. 105–115.

[17] **Ris, F. N.** *Interval analysis and applications to linear algebra.* Ph. D. Thesis, Oxford University, 1972.

[18] **Rump, S.** *Personal communication.* See [2], p. 18.

Received: February 28, 1994
Revised version: July 27, 1994

H. Ratschek
Mathematisches Institut
der Universität Düsseldorf
Germany

J. Rokne
Dept. of Computer Science
The University of Calgary
Canada