Robust State Feedback for Interval Systems: An Interval Analysis Approach*

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Abstract

The problem of robust state feedback design for a linear dynamical system with uncertain (interval) parameters is considered. The designed state feedback controller has place all the coefficients of the closed loop system characteristic polynomial within assigned closed loop interval characteristic polynomial. A condition is derived using certain known facts about matrix minors and its characteristic equation. The derived condition assigns the closed loop coefficients of the system characteristic polynomial within the assigned closed loop interval polynomial, if certain inequalities admit a positive solution. The method is simple and has advantage that it does not require system canonical transformation. The efficacy of the method is illustrated using numerical examples.

Keywords: interval analysis, controller processes, feedback
AMS subject classifications: 65K99, 65G99

1 Introduction

The problem of designing robust controllers for process plants having unknown but bounded parameter uncertainties, which often called the interval process plants either in the form of transfer function or state space model, has received considerable attention [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. Since control systems operate under large uncertainties it is important to study stability robustness in the presence of uncertainty. The uncertainty present in the control system causes degradation of system performance and destabilization. Various analysis and design techniques are essentially meant for application to a "nominal" model. The resulting design is said

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to be robust; if the system performs within acceptable limits in the face of significant parameter variations and model uncertainties. The need to incorporate robustness in design is necessitated by the fact that for most practical system, the model is known only approximately. For example, in modeling a chemical process, there unavoidably exist uncertainties due to poor process knowledge, nonlinearities, unknown internal or external noises, environment influence, time varying parameters, changing operating conditions, etc. Therefore, to build a very accurate model that describes the physical process exactly may be very costly and it may turn out to be impractical from the viewpoint of analysis and design. However, a simplified model may not adequately represent the actual process system and may result in an unacceptable design. Similarly in the aircraft industry, the aircraft model is constructed using the data obtained from the wind-tunnel experiments on the aircraft body. As a consequence, the parameters of the model would not have a specific value; rather they are known to lie within an interval. Since the actual flight data are not available the controller should be able to account for the unmodeled parameters that can be obtained only when the aircraft is airborne. The other examples include robotic manipulators, nuclear reactors, electrical machines and large power networks etc., which have parametric uncertainties for the entire range of operation. An important approach to this subject is via expressing the characteristic polynomial by an interval polynomial, i.e. a polynomial whose coefficient each varies independently in a prescribed interval. The stability analysis of polynomials subjected to parameter uncertainty have received considerable attention after the celebrated theorem of Kharitonov [18], which assures robust stability under the condition that four specially constructed extreme polynomials, called Kharitonov polynomials are Hurwitz. The most of the authors used model description by interval transfer function [1, 3, 5, 7, 18, 19]. The model P regulator synthesis for state space model having interval parameters has been considered in several work [6, 8, 9, 11, 12, 20, 21, 22, 23] . In this paper, a method is presented to design a model P regulator (a robust state feedback controller) which has to place all the coefficients of the system's closed loop characteristic polynomial within assigned closed loop interval characteristic polynomial. A condition is derived using some known facts about matrix minors and its characteristic equation. The derived condition assigns the closed loop coefficients of the system characteristic polynomial within the assigned closed loop interval polynomial if certain inequalities admit a positive definite solution. This method is simple as compared to method presented in [11, 12]. The proposed method does not require system canonical transformation as required in [8, 9]. The paper is organized as: Section 2 gives introduction to interval arithmetic analysis and matrix theory preliminaries. Section 3 describes the problem of robust states feedback design for interval systems. In Section 4 main results are presented which are utilized to design a robust state feedback for interval system. In Section 5, numerical examples are illustrated to show the efficacy of the proposed method for model P regulator design for interval plant in state space. Finally conclusion is given in Section 6.

2 Interval analysis and matrix theory preliminaries

Throughout remaining part of the paper we will use standardized notations in interval analysis [24]. The bold font will denote interval values, whereas usual font will denote real (i.e. non-interval) values. Underlining and overlining an interval will denote the lower and upper bounds of an interval respectively. An interval number $[\underline{x}, \overline{x}]$ can be defined by the set of $\mathbf{x} \subset \mathbb{R}$ (the reals) such that $\underline{x} \leq x \leq \overline{x}$. For $\underline{x} = \overline{x}$, the interval number becomes $[\underline{x}, \overline{x}]$ which can be described as a degenerate interval. The arithmetic operations on intervals are defined as follows: [25, 26, 27, 28, 29, 30, 31]

1. $[\underline{x}, \overline{x}] + [\underline{y}, \overline{y}] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$
2. $[\underline{x}, \overline{x}] \times [\underline{y}, \overline{y}] = [\min(\underline{xy}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{xy}), \max(\underline{xy}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{xy})]$
3. $[\underline{x}, \overline{x}] - [\underline{y}, \overline{y}] = [\underline{x} - \overline{y}, \overline{x} - \underline{y}]$
4. $[\underline{x}, \overline{x}] \div [\underline{y}, \overline{y}] = [\underline{x}, \overline{x}] \times \left[\frac{1}{\overline{y}}, \frac{1}{\underline{y}}\right]$ provided that, $0 \notin [\underline{y}, \overline{y}]$.
5. $$\text{width}\,\boldsymbol{x} = \overline{x} - \underline{x} \geq 0;$$
   its *radius* is
6. $$\text{rad}\,\boldsymbol{x} = \frac{1}{2}\text{width}\,\boldsymbol{x} = \frac{1}{2}(\overline{x} - \underline{x}),$$
   and its *midpoint* is
7. $$\text{mid}\,\boldsymbol{x} = \frac{1}{2}(\overline{x} + \underline{x}).$$

Alternatively, the interval number $\boldsymbol{x}$ can be represented as $\boldsymbol{x} = [\underline{x}, \overline{x}] = \{x \in \boldsymbol{x}$, iff $\underline{x} \leq x \leq \overline{x}\}= [x_0 - \Delta x, x_0 + \Delta x]$, where $x_0 = (\underline{x} + \overline{x})/2$ (the nominal value) and $\Delta x = (\overline{x} - \underline{x})/2$ (the uncertainty).
An interval matrix by definition [2] is a real matrix in which all the elements are known only to the extent that each element belongs to a specified interval. For all $n \times n$ interval real matrices, $\boldsymbol{F} = \{\boldsymbol{f}_{ij}\} \in \mathbb{IR}^{n\times n}$ with interval elements $\boldsymbol{f}_{ij}$, and $\boldsymbol{G} = \{\boldsymbol{g}_{ij}\} \in \mathbb{IR}^{n\times n}$ with interval elements $\boldsymbol{g}_{ij}$, for all i and j, the addition, subtraction and multiplication operations can be written as follows:

1. $\boldsymbol{F} \pm \boldsymbol{G} = \{\boldsymbol{f}_{ij} \pm \boldsymbol{g}_{ij}\} \in \mathbb{IR}^{n\times n}$,
2. $\boldsymbol{F} * \boldsymbol{G} = \{\boldsymbol{f}_{ij}\}\{\boldsymbol{g}_{ij}\} = \left\{\sum_{k=1}^{n} \boldsymbol{f}_{ik} \times \boldsymbol{g}_{kj}\right\} \in \mathbb{IR}^{n\times n}$.

Preliminaries of Matrix Theory:
Consider square matrix $A \in \mathbb{R}^{n\times n}$,

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \tag{1}$$

an $r \times r$ principal submatrix of A is a submatrix that lies on the same set of r rows and columns, and $r \times r$ principal minor is determinant of an $r \times r$ principal submatrix. In other words,$r \times r$ principal minors are obtained by deleting the same set of $n - r$ rows and columns, and there are $\binom{n}{r} = n!/r!(n-r)!$ such minors.
The characteristic polynomial of the matrix A in equation (1) can be written as [32],

$$\Delta(\lambda) = \lambda^n - s_1\lambda^{n-1} + s_2\lambda^{n-2} + ... + (-1)^j s_j \lambda^{n-j} + ... + (-1)^n s_n$$

where s_j is sum of principal minors of order j i.e

1. $s_1 = a_{11} + a_{22} + ... + a_{nn}$, where $a_{11}, a_{22}, ..., a_{nn}$ are first order principal minors of matrix A

2. $s_2 = p_{21} + p_{22} + ... + p_{2n}$, where p_{21}, p_{22},$\cdots$,p_{2n} are second order principal minors of matrix A
   $\vdots$
3. $s_j = p_{j1} + p_{j2} + ... + p_{jn}$, where p_{j1}, p_{j2},$\cdots$,p_{jn} are j^{th} order principal minors of matrix A
4. $s_n = |A|$ where $|A|$ is n^{th} order principal minor of matrix A

3 Robust state feedback design

Consider a linear MIMO uncertain (Interval) system described by the state space equation as

$$\begin{aligned} \dot{x} &= \boldsymbol{A}x + \boldsymbol{B}u \\ y &= \boldsymbol{C}x \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$, $\boldsymbol{B} \in \mathbb{IR}^{n\times m}$, $\boldsymbol{C} \in \mathbb{IR}^{p\times n}$. The entries of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are unknown but bounded in given compact set ; i.e. $\boldsymbol{A} = [\underline{\boldsymbol{A}}, \overline{\boldsymbol{A}}]$, $\boldsymbol{B} = [\underline{\boldsymbol{B}}, \overline{\boldsymbol{B}}]$, $\boldsymbol{C} = [\underline{\boldsymbol{C}}, \overline{\boldsymbol{C}}]$ are interval system matrix, input matrix, and output matrix with elements lying in known upper and lower bound respectively. It is also assumed that the pair $\boldsymbol{A}, \boldsymbol{B}$ is controllable in the sense of definition [12]. Let the robust model P regulator or the linear state feedback control law be,

$$u = Kx \tag{3}$$

Let the closed loop characteristics polynomial of system (2) under state feedback in (3) be,

$$\boldsymbol{\Delta}(\lambda) = \lambda^n - \boldsymbol{s_1}\lambda^{n-1} + \boldsymbol{s_2}\lambda^{n-2} + ... + (-1)^n \boldsymbol{s_n} \tag{4}$$

where $\boldsymbol{s}_i$ is coefficient of the equation (4) such that $\boldsymbol{s}_i = [\underline{s}_i, \overline{s}_i]$ ($i = 1, 2, ..., n$)
Consider a desired (or target) interval characteristic polynomial as per desired specifications. The well known Kharitonov theorem [18] may be used to construct an asymptotically stable interval polynomial. This form of specification is more natural to designer because generally he does not know how to choose closed loop characteristic roots but he has good idea about desired region (or target region).

$$[\boldsymbol{d}(s)] = s^n + \boldsymbol{d}_1 s^{n-1} + \boldsymbol{d}_2 s^{n-2} + \cdots + \boldsymbol{d}_n \tag{5}$$

where $\boldsymbol{d}_i$ is coefficient of the equation (5) such that $\boldsymbol{d}_i = [\underline{d}_i, \overline{d}_i]$ ($i = 1, 2, ..., n$).

Definition 1 *An interval system ($\boldsymbol{A}$,$\boldsymbol{B}$) is said to be stabilizable if there exist a linear state feedback control law $u = Kx$ with $K \in \mathbb{R}^n$ such that characteristic polynomial of closed loop system*

$$\boldsymbol{\Delta}(\lambda) = det(sI - \boldsymbol{A} - \boldsymbol{B}K) \tag{6}$$

is a Hurwitz invariant polynomial i.e all the roots of uncertain polynomial (6) are in the are strict left half of the complex plane.

Definition 2 *An interval system ($\boldsymbol{A}$,$\boldsymbol{B}$) is said to be controllability invariant if the pair ($\boldsymbol{A}$,$\boldsymbol{B}$) is controllable in usual sense for any fixed value of uncertain parameters, that is*

$$rank[A - sI, B] = n \tag{7}$$

for every $a_{ij} \in [\underline{a}_{ij}, \overline{a}_{ij}], b_{ij} \in [\underline{b}_{ij}, \overline{b}_{ij}]$

The model P regulator or state feedback design problem can be stated as:
Design a robust state feedback control as in (3) to find a real $m \times n$ matrix K satisfying the inclusions

$$\det(sI - \boldsymbol{A} - \boldsymbol{b}K) \subseteq [\boldsymbol{d}(s)] \quad (8)$$

for every $A \in \boldsymbol{A}$, $B \in \boldsymbol{B}$, where $[\boldsymbol{d}(s)]$ is an assigned asymptotically stable interval polynomial.

4 Main results

For the given problem we consider two cases: $m = 1$, $m \geq 2$.
Case $m = 1$ (SISO Case): Suppose that $\boldsymbol{B} = \boldsymbol{b}$ is a column vector, $K = k$ is a row vector. The following definition is based on the results of [12]

Definition 3 *The pair $(\boldsymbol{A}, \boldsymbol{b})$ is controllable for any $A \in \boldsymbol{A}, b \in \boldsymbol{b}$ if a square interval matrix*

$$\boldsymbol{Y} = [\boldsymbol{b}, \boldsymbol{A} * \boldsymbol{b}, ..., \boldsymbol{A}^{n-1} * \boldsymbol{b}] \quad (9)$$

satisfies the condition

$$0 \notin Det[\boldsymbol{Y}] \quad (10)$$

where * represents interval multiplication and Det[.] denotes an interval extension of the function det[.] [25]. The state feedback problem for the SISO case can be stated as: Design a robust state feedback control as in (3) to find a real $1 \times n$ matrix k satisfying the inclusions

$$\det(sI - \boldsymbol{A} - \boldsymbol{b}k) \subseteq [\boldsymbol{d}(s)] \quad (11)$$

for every $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, where $[\boldsymbol{d}(s)]$ is an assigned asymptotically stable interval polynomial. The condition given in the following theorem provides a solution to model P regulator (robust state feedback control) problem.

Theorem 1 *The control law $u = kx$ stabilizes system ($\boldsymbol{A}$, $\boldsymbol{b}$) and robustly assigns the closed loop poles in prescribed region as described by (11) if and only if the following inequalities admit a positive solution*

$$\underline{d}_1 \leq \boldsymbol{s}_1(\boldsymbol{A}) + \boldsymbol{b}_i k_i \leq \overline{d}_1 \quad (12)$$

$$\underline{d}_j \leq \boldsymbol{s}_j(\boldsymbol{A}) + \boldsymbol{s}_j(\boldsymbol{A}, i)_{\boldsymbol{b}} k_i \leq \overline{d}_j \quad (13)$$

where $i = 1, 2, 3, ...n$, $j = 2, 3, ...n$ and $\boldsymbol{s}_j(\boldsymbol{A}, i)_{\boldsymbol{b}}$ is sum of j^{th} order principal minors of matrix $\boldsymbol{A}$ which includes elements of vector '$\boldsymbol{b}'$ as its i^{th} column.

Proof 1 *Consider system matrix $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ and input vector $\boldsymbol{b} \in \mathbb{IR}^{n\times 1}$ and state feedback vector $k \in \mathbb{R}^{1\times n}$*

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_{11} & \cdots & \boldsymbol{a}_{1n} \\ \vdots & \ddots & \vdots \\ \boldsymbol{a}_{n1} & \cdots & \boldsymbol{a}_{nn} \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \vdots \\ \boldsymbol{b}_n \end{bmatrix} \quad (14)$$

where $a_{ij} = [\underline{a}_{ij}, \overline{a}_{ij}]$ and $b_i = [\underline{b}_i, \overline{b}_i]$ with $i, j = 1, 2, ...n$

$$k = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} \quad (15)$$

$$\boldsymbol{A}+\boldsymbol{b}k=\begin{bmatrix}\boldsymbol{a}_{11}+\boldsymbol{b}_1k_1 & \cdots & \boldsymbol{a}_{1n}+\boldsymbol{b}_1k_n\\ \vdots & \ddots & \vdots\\ \boldsymbol{a}_{n1}+\boldsymbol{b}_nk_1 & \cdots & \boldsymbol{a}_{nn}+\boldsymbol{b}_nk_n\end{bmatrix} \tag{16}$$

The interval arithmetic [27] is used for the following computations.
s_1 *= Sum of first order principal minor of* $\boldsymbol{A}+\boldsymbol{b}k$ *therefore,*

$$\boldsymbol{s}_1=(\boldsymbol{a}_{11}+\boldsymbol{b}_1k_1)+(\boldsymbol{a}_{22}+\boldsymbol{b}_2k_2)+\cdots+(\boldsymbol{a}_{nn}+\boldsymbol{b}_nk_n) \tag{17}$$

This can be rewritten as

$$\boldsymbol{s}_1=\boldsymbol{s}_1(\boldsymbol{A})+\boldsymbol{b}_ik_i \tag{18}$$

$\boldsymbol{s}_2$ *= Sum of second order principal minor of* $\boldsymbol{A}+\boldsymbol{b}k$

$$\boldsymbol{s}_2=\begin{vmatrix}\boldsymbol{a}_{11}+\boldsymbol{b}_1k_1 & \boldsymbol{a}_{12}+\boldsymbol{b}_1k_2\\ \boldsymbol{a}_{21}+\boldsymbol{b}_2k_1 & \boldsymbol{a}_{22}+\boldsymbol{b}_2k_2\end{vmatrix}+\begin{vmatrix}\boldsymbol{a}_{11}+\boldsymbol{b}_1k_1 & \boldsymbol{a}_{13}+\boldsymbol{b}_1k_3\\ \boldsymbol{a}_{31}+\boldsymbol{b}_3k_1 & \boldsymbol{a}_{33}+\boldsymbol{b}_3k_3\end{vmatrix}+\cdots+$$

$$\begin{vmatrix}\boldsymbol{a}_{11}+\boldsymbol{b}_1k_1 & \boldsymbol{a}_{1n}+\boldsymbol{b}_1k_n\\ \boldsymbol{a}_{31}+\boldsymbol{b}_3k_1 & \boldsymbol{a}_{nn}+\boldsymbol{b}_nk_n\end{vmatrix}+\cdots+\begin{vmatrix}\boldsymbol{a}_{22}+\boldsymbol{b}_2k_2 & \boldsymbol{a}_{23}+\boldsymbol{b}_2k_3\\ \boldsymbol{a}_{32}+\boldsymbol{b}_3k_2 & \boldsymbol{a}_{33}+\boldsymbol{b}_3k_3\end{vmatrix}+$$

$$\begin{vmatrix}\boldsymbol{a}_{22}+\boldsymbol{b}_2k_2 & \boldsymbol{a}_{2n}+\boldsymbol{b}_2k_n\\ \boldsymbol{a}_{n2}+\boldsymbol{b}_3k_2 & \boldsymbol{a}_{nn}+\boldsymbol{b}_nk_n\end{vmatrix}+\cdots+$$

$$\begin{vmatrix}\boldsymbol{a}_{n-1,n-1}+\boldsymbol{b}_{n-1}k_{n-1} & \boldsymbol{a}_{n-1,n}+\boldsymbol{b}_{n-1}k_n\\ \boldsymbol{a}_{n,n-1}+\boldsymbol{b}_nk_{n-1} & \boldsymbol{a}_{nn}+\boldsymbol{b}_nk_n\end{vmatrix}$$

This can be rewritten as,

$$\boldsymbol{s}_2=\boldsymbol{s}_2(\boldsymbol{A})+\boldsymbol{s}_2(\boldsymbol{A},i)_{\boldsymbol{b}}k_i \tag{19}$$

where $i=1,2,3...n$ *and* $\boldsymbol{s}_2(\boldsymbol{A})$ *is sum of second order principal minors of matrix* $\boldsymbol{A}$ *and* $\boldsymbol{s}_2(\boldsymbol{A},j)_{\boldsymbol{b}}$ *is sum of second order principal minors of matrix* $\boldsymbol{A}$ *which includes elements of vector b with* i^{th} *column is replaced by vector* $\boldsymbol{b}$*. Therefore for* n^{th} *order principal minor, equation (19) can be written as*

$$\boldsymbol{s}_n=s_n(\boldsymbol{A})+\boldsymbol{s}_n(\boldsymbol{A},i)_{\boldsymbol{b}}k_i \tag{20}$$

where $i=1,2,3...n$*. In general for* j^{th} *order principal minor equation (20) can be rewritten as*

$$\boldsymbol{s}_j=\boldsymbol{s}_j(\boldsymbol{A})+\boldsymbol{s}_j(\boldsymbol{A},i)_{\boldsymbol{b}}k_i \tag{21}$$

where $i=1,2,3...n, j=2,3,...n$ *and* $\boldsymbol{s}_j(\boldsymbol{A})$ *is sum of* j^{th} *order order principal minors of matrix* $\boldsymbol{A}$ *and* $\boldsymbol{s}_j(\boldsymbol{A},i)_{\boldsymbol{b}}$ *is sum of* j^{th} *order principal minors of matrix* $\boldsymbol{A}$ *which includes elements of vector* $\boldsymbol{b}$ *with* i^{th} *column is replaced by vector* $\boldsymbol{b}$*. Hence condition (12) and (13) can be obtained by comparing equation (4) and (5) and Mapping theorem [2], which proves the theorem (1)*

Case $m\geq 2$ (MIMO Case):

Definition 4 *The pair* $(\boldsymbol{A},\boldsymbol{B})$ *is controllable for any* $A\in\boldsymbol{A}$ *and* $B\in\boldsymbol{B}$ *if and only if the interval controllability matrix [12]*

$$\boldsymbol{Y}=[\boldsymbol{B},\boldsymbol{A}*\boldsymbol{B},...\boldsymbol{A}^{n-1}*\boldsymbol{B}] \tag{22}$$

satisfies the equation

$$rank\boldsymbol{Y}=n \tag{23}$$

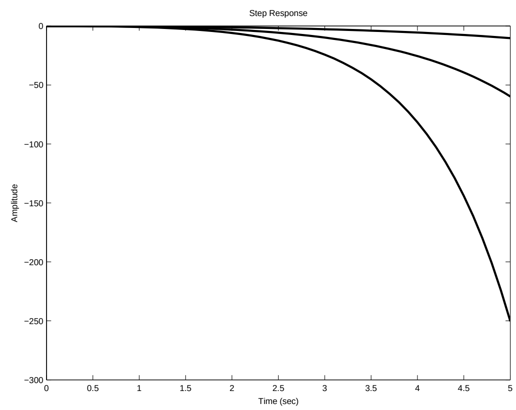

Figure 1: Example 1: Step response of extreme plants for the open loop system with parameters from the intervals

Definition 5 *The pair* $(\boldsymbol{A},\boldsymbol{B})$ *is controllable for any* $A \in \boldsymbol{A}$ *and* $B \in \boldsymbol{B}$ *if [12]*

$$0 \notin Det[\boldsymbol{Y}] \tag{24}$$

where Det[.] denotes an interval extension of the function det[.] [25]. Suppose that for any $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$ the pair $(\boldsymbol{A},\boldsymbol{B})$ is controllable. Moreover assume that for any $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$ the pair $(\boldsymbol{A},\boldsymbol{B})$ is cyclic pair [33]. Then we can (almost always) find a real m vector q which guarantees the controllability of the pair $(\boldsymbol{A},\boldsymbol{B}*q)$, i.e. the $n \times n$ interval controllability matrix $\boldsymbol{Y}1 = [\boldsymbol{B}*q, \boldsymbol{A}*\boldsymbol{B}*q, ..., \boldsymbol{A}^{n-1}*\boldsymbol{B}*q]$ satisfies the condition (24). Considering $\boldsymbol{b} = \boldsymbol{B}*q$ and using the Theorem (1) we can calculate a real row vector k . Then the $m \times n$ real matrix K results from the formula

$$K = qk \tag{25}$$

In conclusion the following algorithm can be used to design a robust state feedback for interval systems.
Step 1. Analyze the controllability of the pair $(\boldsymbol{A},\boldsymbol{B})$ for all $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$. If this pair is not controllable then the problem has no solution.
Step 2. If $m = 1$ then go to step 3 otherwise chose some real numbers as the elements of m vector q. Calculate $\boldsymbol{b} = \boldsymbol{B}*q$. If the pair $(\boldsymbol{A},\boldsymbol{B}*q)$ is controllable then go to step 3 else chose another vector q.
Step 3. Determine the state feedback controller k by using theorem (1).
Step 4. If $m = 1$ then $K = k$. If $m \geq 2$ then $K = qk$.

5 Numerical Examples

Example 1: Consider the SISO interval plant described in state space as [10],

$$\boldsymbol{A} = \begin{bmatrix} [-0.5, 0.5] & [0.5, 1] \\ [0.0, 0.0] & [0.5, 1] \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} [0.0, 0.0] \\ [-1, -0.9] \end{bmatrix} \tag{26}$$

It is necessary to find row vector $k = [k_1, k_2]$ so that for every polynomial coefficient of closed loop system are located inside the interval coefficient of interval polynomial

$$\boldsymbol{d}(s) = s^2 + [1,5]s + [1, 11.5] \tag{27}$$

By applying conditions (12), (13) in theorem (1), we obtain the following inequalities. Hence we get a Linear programming (LP) problem to solve following inequalities for $k = [k_1, k_2]$ subject to minimization of performance index $f(k_1, k_2) = \sum k_i$

$$\begin{aligned} -5 \le -0.5 + 0.5 - k_2 \le -1 \\ -5 \le 0.5 + 1 - 0.9k_2 \le -1 \\ 1 \le -0.5 - 0.5k_2 + 0.45k_1 \le 11.5 \\ 1 \le 0.5 + 0.5k_2 + k_1 \le 11.5 \end{aligned} \tag{28}$$

By solving above linear programming problem by using MATLAB [34], we obtained the controller parameters as $k_1 = 8.6$, $k_2 = 4$. The step response for open loop and closed loop system are plotted in Fig. 1 and Fig. 2 respectively. From the step response it is evident that obtained controller is robust against the parameter variation. The Nyquist plot for extreme and middle plants is shown in Fig. 3, which shows the stability of the closed loop system.

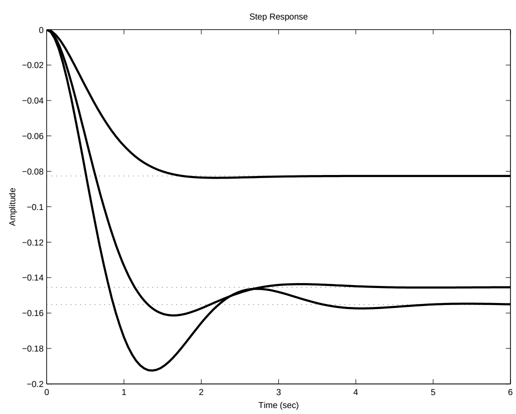

Figure 2: Example 1: Step response of extreme plants for the closed loop system with parameters from the intervals

Example 2: Consider a stabilization control problem [12] for a helicopter longitudinal motion speed; the helicopter longitudinal motion can be described by linear dynamical state-space model (2) with $n = 3$, $m = 2$ and the matrices,

$$\boldsymbol{A} = \begin{bmatrix} [-0.031, -0.0128] & [-3.4, -0.1] & [-9.8, -9.8] \\ [-0.00077, 0.0007] & [-0.32, -0.31] & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{29}$$

$$\boldsymbol{b} = \begin{bmatrix} [-18, -15] & 0 \\ 0 & [-3.3, -3] \\ 0 & 0 \end{bmatrix} \tag{30}$$

In the vector $x = [x_1, x_2, x_3]^T$ x_1 is a deviation of the longitudinal motion projection, x_2 is an angular speed deviation, x_3 is a pitch angular deviation.

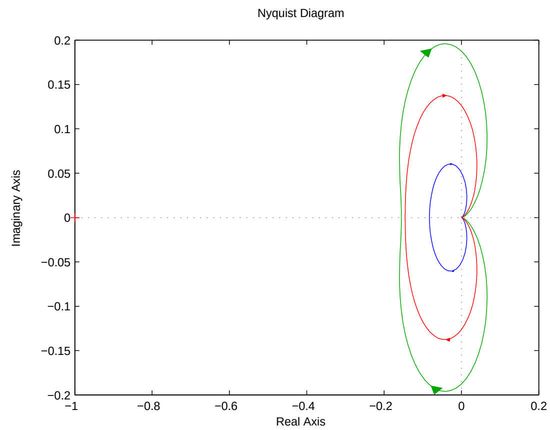

Figure 3: Example 1: Nyquist plot for extreme and middle plants system with parameters from the intervals

The bound on eigenvalues of the interval matrix can be obtained using the method suggested in [14], [15], [16] and [13]. The bounds on the open loop eigenvalues obtained using the method described in [13] are given below in Table (1). It is necessary to find a real 2×3 matrix K such that for every real

Eigenvalue number	Lower bound	Intermediate bound	Upper bound
1	$-0.2351 + 0.0898i$	$-0.2283 + 0.0901i$	$-0.2215 + 0.0898i$
2	$-0.2351 - 0.0898i$	$-0.2283 - 0.0901i$	$-0.2215 - 0.0898i$
3	0.1192	0.1196	0.1201

Table 1: Open loop eigenvalues of the system

$A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$ the characteristic polynomial coefficients of the closed-loop matrix $\boldsymbol{A} + \boldsymbol{B}K$ are located within the interval coefficients of the given interval stable polynomial

$$[\boldsymbol{d}(s)] = s^3 + [3,4]s^2 + [2,8]s + [0.5, 5.5]. \tag{31}$$

The root location of $\boldsymbol{d}(s)$ is shown in Fig. 4. The controllability analysis of $(\boldsymbol{A}, \boldsymbol{B})$ shows that this pair is controllable for all $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$. To establish robust controllability of the pair $(\boldsymbol{A}, \boldsymbol{B})$ the method in [12] is used. We can also use the method suggested in [35] to establish robust controllability of the pair $(\boldsymbol{A}, \boldsymbol{B})$.

We chose arbitrarily $q = (0.8, 1.2)^T$ and compute the vector

$$\boldsymbol{b} = \boldsymbol{B} * q = \begin{bmatrix} [-14.4, -12] \\ [-3.96, -3.6] \\ 0 \end{bmatrix} \tag{32}$$

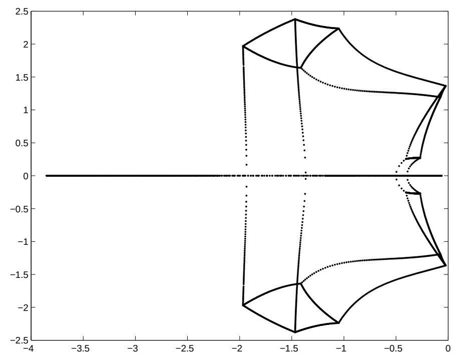

Figure 4: Example 2: Root cluster of $d(s)$

The pair $(\boldsymbol{A}, \boldsymbol{b})$ is controllable because the condition from (24) is satisfied: $0 \notin \mathrm{Det}(\boldsymbol{b}, \boldsymbol{A} * \boldsymbol{b}, \boldsymbol{A}^2 * \boldsymbol{b})$.
By applying condition (12), (13), we get inequalities as given below. Therefore we have LP problem to solve following inequalities for $k = [k_1, k_2, k_3]$ subject to minimization of performance index $f(k_1, k_2, k_3) = \sum k_i$ to satisfy the requirement $\Re(\Delta(\lambda)) \subseteq \Re(d(s))$

$$\begin{aligned} -4 \leq -0.351 - 14.4k_1 - 3.96k_2 \leq -3 \\ -4 \leq -0.3228 - 12k_1 - 3.6k_2 \leq -3 \\ 2 \leq 0.0011 - 9.7441k_1 + 0.0348k_2 + 3.6k_3 \leq 8 \\ 2 \leq 0.0101 + 4.2481k_1 + 0.1146k_2 + 3.96k_3 \leq 8 \\ -5.5 \leq 35.28k_1 - 0.1146k_3 + 0.0068 \leq -0.5 \\ -5.5 \leq 38.8081k_1 - 0.0348k_3 + 0.0076 \leq -0.5 \end{aligned} \tag{33}$$

By solving above linear programming problem by using MATLAB [34] we obtained the controller parameters as $k_1 = -0.0181$, $k_2 = 0.8069$, $k_3 = 0.5011$.
Then the state feedback gain matrix of modal P-regulator can be computed as

$$K = q * k = \begin{bmatrix} -0.0145 & 0.6455 & 0.4009 \\ -0.0218 & 0.9683 & 0.6014 \end{bmatrix} \tag{34}$$

The state feedback controller, when applied to the system, results in a stable closed loop system. The closed loop eigenvalues of closed loop system are given in Table (2).

Eigenvalue number	Lower bound	Intermediate bound	Upper bound
1	-2.5354	-2.4453	-2.3686
2	$-0.3968 + 0.3610i$	$-0.3512 + 0.4053i$	$-0.2990 + 0.4440i$
3	$-0.3968 - 0.3610i$	$-0.3512 - 0.4053i$	$-0.2990 - 0.4440i$

Table 2: Closed loop eigenvalues of the system

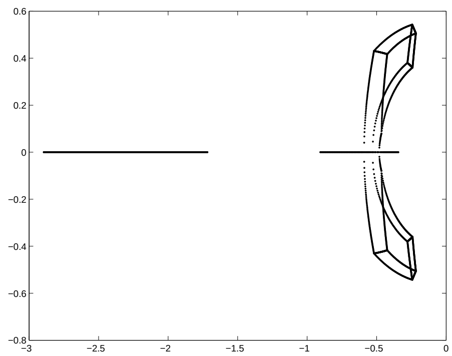

Figure 5: Example 2: Root cluster of $\Delta(\lambda) = \mathrm{Det}(sI - \mathbf{A} - \mathbf{b}K)$

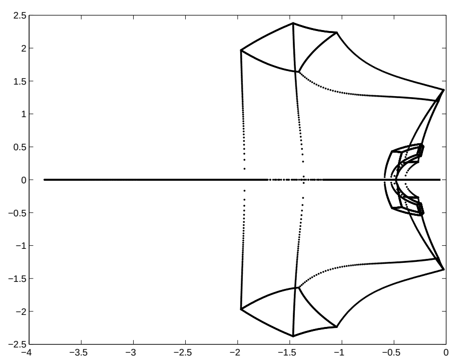

Figure 6: Root cluster of $d(s)$ and $\Delta(\lambda) = \mathrm{Det}(sI - \mathbf{A} - \mathbf{b}K)$

With this controller we construct the root space of $\Re(\Delta(\lambda))$ which is shown in Fig. 5. Fig. 6 also shows that roots space $\Re(\Delta(\lambda))$ is clearly contained in $\Re(\boldsymbol{d}(s))$. The step response of the open loop and closed loop systems are shown in Fig. 7, and Fig. 8 respectively. From the step response of closed loop system it is evident that system has been stabilized for all possible elements from the interval.

6 Conclusion

The problem of Robust state feedback design for linear dynamical system with uncertain (interval) parameters is considered. The designed state feedback controller has to place all the coefficients of the closed loop system characteristic polynomial within assigned closed loop interval characteristic polynomial. A condition is derived using some known facts about matrix minors and its characteristic equation. The derived condition assigns the closed loop coefficients of the system characteristic polynomial within the assigned closed loop inter-

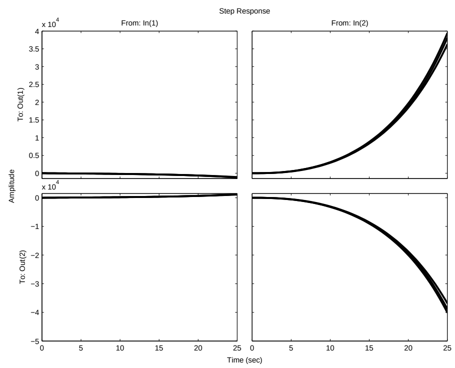

Figure 7: Example 2 : Step Response of open loop system with parameters from interval

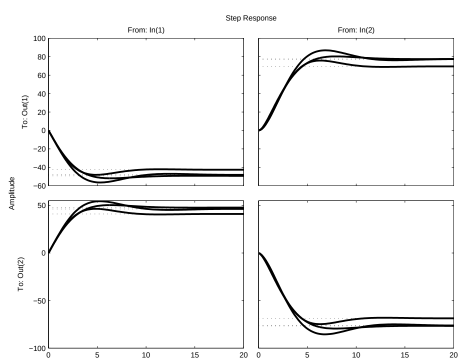

Figure 8: Example 2: Step Response of closed loop system with parameters from interval

val polynomial, if certain inequalities admit a positive solution. The method is simple and has advantage that it does not require system canonical transformation. The efficacy of the method is illustrated using numerical examples. The designed state feedback is robust as evident from the simulation results for the entire range of parameter variation within the given known bounds.

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