Dynamics with a Range of Choice*

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Abstract

Mathematical setting for discrete dynamics is a state space, X, and a map $S : X \to X$ (the evolution operator) which defines the change of a state over one time step. Dynamics with choice, as we define it in [2], is a generalization of discrete dynamics where at every time step there is not one but several available maps that can transform the current state of the system. Many real life processes, from autocatalytic reaction systems to switched systems to cellular biochemical processes, exhibit the properties described by dynamics with choice. We are interested in the long-term behavior of such systems. In [2] we studied dynamics with choice with a finite number of available maps, $S_0, S_1, \ldots, S_{N-1}$. The orbit of a point $x \in X$ then depends on the infinite sequence of symbols from the set $\mathcal{J} = \{0, 1, \ldots, N-1\}$ encoding the order of maps S_j applied at each step. Denote by Σ the space of all one-sided infinite sequences of symbols from $\mathcal{J}$ and denote by σ the shift operator that erases the first symbol in sequences. We define the dynamics on the state space X with the choice of the maps $S_0, S_1, \ldots, S_{N-1}$ as the discrete dynamics on the state space $\mathfrak{X} = X \times \Sigma$ with the evolution operator $\mathfrak{S} : (x, w) \mapsto (S_{w(0)}(x), \sigma(w))$, where $w(0)$ is the first symbol in the sequence w.

In this paper we address the case when there is possibly a continuum of available maps parameterized by points from the interval $[0, 1]$ or any metric compact $\mathcal{J}$. (Think of a system of equations with parameters, where each parameter may change from step to step while staying within a prescribed interval.) We say that there is *a range of choice*. We give mathematical description of dynamics with a range of choice and prove general results on the existence and properties of global compact attractors in such dynamics. One of practical consequences of our results is that when the parameters of a nonlinear discrete-time system are not known exactly and/or are subject to change due to internal instability, or a strategy, the long term behavior of the system may not be correctly described by a system with "averaged" values for the parameters. There may be a Gestalt effect.

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1 Introduction

In many real-life and engineered systems, the structural parameters are known to lie within certain intervals but often their exact values are not known. Moreover, often there are no exact values of the parameters because the parameters may change in time due to internal instability or due to exterior forces (e.g., the system is not closed and/or we intend to control it). There are many possible ways to mathematically describe a deterministic discrete-time dynamics in such circumstances. We will concentrate on two descriptions. Let X be the state space and let $\mathcal{J}$ be the parameter space of the system ($\mathcal{J}$ may be finite or infinite; e.g., $\mathcal{J}$ may be a product of intervals). For $j \in \mathcal{J}$, denote by S_j the one-step transformation map $x \mapsto S_j(x)$ that describes the one time step change of the states of the system with (exact) parameters j. If the value $j \in \mathcal{J}$ is unpredictable, it makes sense to follow the *interval analysis* approach and assign to x the whole "interval" of (possible) states $\cup_{j\in\mathcal{J}} S_j(x)$. Immediately we are led to look at the dynamics of *subsets* of X generated by the map $F : A \mapsto F(A) = \bigcup_{j\in\mathcal{J}} S_j(A)$. This dynamics is also studied under the name of *iterated function systems*, IFS for short, [3, 4]. In the context of IFS, the map F is associated with the names of Hutchinson and Barnsley.

Another approach was suggested by the authors in [9], we called it *dynamics with choice*. To record which maps S_j and in what order are applied, we use finite or infinite sequences of elements of $\mathcal{J}$. Borrowing terminology from dynamical systems, $\mathcal{J}$ is the alphabet, the elements of $\mathcal{J}$ are symbols, the finite or infinite sequences of symbols are (finite or infinite) words, or strings. Denote by $\Sigma_{\mathcal{J}}$ the space of all one-sided infinite strings of symbols from $\mathcal{J}$. The elements of $\Sigma_{\mathcal{J}}$ can be viewed as *strategies* or *plans*, because the symbols and their order in a word $w \in \Sigma_{\mathcal{J}}$ tell us which maps S_j and in what order are applied. We identify the strings from $\Sigma_{\mathcal{J}}$ with maps from the (semigroup of) non-negative integers, $\mathbb{Z}_{\geq 0}$, into $\mathcal{J}$; thus, for $w : \mathbb{Z}_{\geq 0} \to \mathcal{J}$ in $\Sigma_{\mathcal{J}}$, we write it as an infinite word $w = w(0)w(1)w(2)\dots$. The shift operator $\sigma : \Sigma_{\mathcal{J}} \to \Sigma_{\mathcal{J}}$ maps w to $\sigma(w)$ so that $\sigma(w)(n) = w(n+1)$; in other words, $\sigma(w) = w(1)w(2)\dots$. Given the state space X and the family of maps S_j, $j \in \mathcal{J}$, we define the corresponding *dynamics with choice* as the dynamics on the product $\mathfrak{X} = X \times \Sigma_{\mathcal{J}}$ generated by the iterations of the map

$$\mathfrak{S} : (x, w) \mapsto (S_{w(0)}(x), \sigma(w)) . \tag{1}$$

In other words, we view the dynamics $x_{n+1} = S_{w(n)}(x_n)$ as a non-autonomous system and use the skew-product (semi)flow approach (see [17]) to describe it.

In [9], we studied the long-term regimes in dynamics with choice in the case when $\mathcal{J}$ is a finite set. In this paper we extend the results of [9] to the case of possibly infinite, compact $\mathcal{J}$, hence the name: dynamics with a *range* of choice. More specifically, we are interested in systems that possess global compact attractors. We give very general sufficient conditions for the existence of a global compact attractor both in the "interval" or IFS dynamics and in the dynamics with a range of choice. We show how those attractors are related. We describe a *restricted dynamics with choice* which corresponds to a restriction of $\Sigma_{\mathcal{J}}$ to a closed subshift, and study attractors for such systems. It is also possible to obtain attractors for individual fixed strategies w. One may expect that the union of such individual attractors when w runs through $\Sigma_{\mathcal{J}}$ should fill the attractor of the whole system. In some cases this is true. However, in general the union is strictly smaller and not even dense in the full attractor.

There are several practical implications of our study. First, the long-term behavior of a system with averaged parameters reflects only a small part of the full attractor.

Second, even if the long-term behavior were known for every strategy of switching parameters, this may not be enough to recover the full attractor (we call this the Gestalt effect).

2 Basic assumptions

Throughout the paper we assume that X is a complete metric space with metric d_X, and $\mathcal{J}$ is a compact metric space with metric $d_{\mathcal{J}}$. The maps $S_j : X \to X$, $j \in \mathcal{J}$, are continuous and bounded (map bounded sets to bounded sets). We equip the space $\Sigma_{\mathcal{J}}$ (of one-sided infinite strings of symbols from alphabet $\mathcal{J}$) with the metric

$$d_\Sigma(w, s) = \sum_{j=1}^{\infty} 2^{-j}\, d_{\mathcal{J}}(w(j), s(j))\,.$$

The space $\Sigma_{\mathcal{J}}$ with the metric d_Σ is compact (see [6]). If $\mathcal{J}$ is a finite set, we choose $d_{\mathcal{J}}(i,j) = 1$, if $i \neq j$ and $= 0$ otherwise. Then d_Σ is equivalent to the metric that assigns distance 2^{-m} between the strings u and v with identical first m symbols and $u(m) \neq v(m)$; we used the latter metric in [9]. The shift, $\sigma : \Sigma_{\mathcal{J}} \to \Sigma_{\mathcal{J}}$, is continuous with respect to the metric topology.

We start with proving a result on the existence of a global compact attractor for the dynamics with a range of choice, i.e., for the dynamics on the space $\mathfrak{X} = X \times \Sigma_{\mathcal{J}}$ generated by iterations of the map $\mathfrak{S}$ defined in (1). We equip $\mathfrak{X}$ with the metric $d_{\mathfrak{X}}((x,u),(y,v)) = d_X(x,y) + d_\Sigma(u,v)$, which makes $\mathfrak{X}$ a complete metric space.

If Y is a metric space and Φ is a map from Y to Y, we denote by (Y, Φ) the discrete semidynamical system generated on Y by iterations of Φ. A set $A \subset Y$ is a global compact attractor of (Y, Φ) if 1) A is compact, 2) A attracts every bounded set in Y, i.e.,

$$\vec{dist}(\Phi^n(B), A) \equiv \sup_{y \in \Phi^n(B)} d_Y(y, A) \underset{n \to +\infty}{\longrightarrow} 0\,,$$

for every bounded set B in Y, and, finally, 3) A is the minimal set with properties 1) and 2). For a detailed information regarding the existence and properties of global compact attractors see, e.g., [2, 12, 13, 7, 18]. In [9], we give a concise list of basic facts we use here as well.

We make the following assumptions on the maps S_j.

Assumption 1. *There is a closed, bounded set* $\mathbf{B} \subset X$ *such that for every bounded set* $A \subset X$ *there exists* $m(A) > 0$ *such that*

$$S_{w(n-1)} \circ \cdots \circ S_{w(1)} \circ S_{w(0)}\ (A) \subset \mathbf{B}$$

for every finite word w *of the length* $|w| = n \geq m(A)$.

[In applications $\mathbf{B}$ is usually a closed ball of radius that depends on the parameters of the model. Showing that for different values of the parameters there is a common estimate on the radius is enough to verify Assumption 1.]

Assumption 2. *There exists a measure of noncompactnes* ψ_X *on* X *such that each operator* S_j *is* ψ_X*-condensding.*

That ψ is a measure of noncompactness here means that ψ is a non-negative function on bounded subsets of X such that

(i) $\psi(A) = 0$ if and only if A is relatively compact;

(ii) If $A_1 \subset A_2$, then $\psi(A_1) \le \psi(A_2)$;

(iii) $\psi(A_1 \cup A_2) = \max\{\psi(A_1),\, \psi(A_2)\}$;

(iv) There exists a constant $c(\psi) \ge 0$ such that

$$|\psi(A_1) - \psi(A_2)| \,\le\, c(\psi)\, d_H(A_1, A_2),$$

where d_H is the Hausdorff distance,

$$d_H(A_1, A_2) = \max\{\vec{dist}\,(A_1, A_2),\, \vec{dist}\,(A_2, A_1)\}.$$

Note that property (iv) implies that the measures of noncompactness of a bounded set and its closure are equal:

(v) $\quad \psi(\overline{A}) = \psi(A)$.

For a detailed discussion of measures of noncompactness see [1]. The most popular measures are the Kuratowski measure of noncompactness, α, and the Hausdorf measure of noncompactness, χ. By definition, $\alpha(A)$ is the infimum of $\epsilon > 0$ such that there is an open cover of A by sets of diameter less than ϵ. The Hausdorf measure of noncompactness of A is the infimum of $\epsilon > 0$ such that A has a finite ϵ-net. Both of these measures of noncompactness satisfy the above properties, [1].

Recall that a continuous bounded map $S : X \to X$ is condensing with respect to the measure of noncompactness ψ (we also say that S is ψ-condensing) iff $\psi(S(A)) \le \psi(A)$ for any bounded A, and $\psi(S(A)) < \psi(A)$ if $\psi(A) > 0$ (i.e., if $\overline{A}$ is not compact).

In applications, people usually apply one of the following sufficient conditions to guarantee that a map S is condensing.

- S is a compact map (i.e., $S : X \to X$ is continuous and maps bounded sets into relatively compact sets);
- S is a contraction: $d_X(S(x), S(y)) < \gamma d_X(x, y)$ for some positive $\gamma < 1$ and for all $x, y \in X$; this works with both Kuratowski and Hausdorff measures ψ;
- in the case X is a Banach space, S is a sum of a compact operator and a strict contraction.

Compact S arise, e.g., in the finite-dimensional dynamics described by differential or difference equations, or, in the infinite dimensional case, in dynamics described by parabolic equations. The "compact + contraction" S appear, e.g., in hyperbolic problems with damping. See [12, 13, 18] and references there.

Note, that Assumptions 1 and 2 are the same as in our paper [9]. Now, since we allow here an infinite number of maps S_j, we need an additional assumption concerning their dependence on the parameter j.

Assumption 3. *For any closed, bounded $A \subset X$, the maps S_j, restricted to A, depend uniformly continuously on j. More precisely, given a closed, bounded A, for every $\epsilon > 0$ there is a $\delta > 0$ such that $\sup_{x \in A} d_X(S_i(x), S_j(x)) \le \epsilon$ provided that $d_{\mathcal{J}}(i, j) \le \delta$.*

3 Attractors for dynamics with a range of choice

We will need the following abstract result whose proof may be found, e.g., in [7, 13, 18].

Theorem 1. *Let Y be a complete metric space and let $\Phi : Y \to Y$ be a continuous, bounded map. Assume that Φ is condensing with respect to some measure of noncompactness ψ_Y satisfying the conditions (i), (ii), and (iii). In addition, assume that there is a bounded set $\mathcal{B} \subset Y$ such that, under the iterations of Φ, the trajectory of any bounded, closed set $A \subset Y$ eventually ends up and stays in $\mathcal{B}$, i.e., $\Phi^n(A) \subset \mathcal{B}$ for all sufficiently large n. Then the discrete semidynamical system (Y, Φ) possesses a global compact attractor, $\mathfrak{A}$. The attractor is (unique and) invariant, $\Phi(\mathfrak{A}) = \mathfrak{A}$. It is the largest closed and bounded invariant set. Through every point y_0 of $\mathfrak{A}$ passes a complete trajectory, i.e., there is an infinite two-sided sequence of points $\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots$ in $\mathfrak{A}$ such that $\Phi(y_n) = y_{n+1}$.*

We are going to apply Theorem 1 to prove the existence of the attractor in the dynamics with choice $(\mathfrak{X}, \mathfrak{S})$. Our Assumption 1 gives the absorbing set $\mathcal{B} = \mathbf{B} \times \mathcal{J}$ in $\mathfrak{X}$. It remains to show that the map $\mathfrak{S}$ is condensing with respect to some measure of noncompactness $\psi_{\mathfrak{X}}$. Because the parameter space $\mathcal{J}$ is compact, there is a natural choice for $\psi_{\mathfrak{X}}$, namely,

$$\psi_{\mathfrak{X}}(\mathfrak{C}) = \psi_X(\mathrm{pr}_X(\mathfrak{C}))\,,$$

where $\mathrm{pr}_X(\mathfrak{C}) = \{x \in X \,:\, (x, u) \in \mathfrak{C},\ \text{for some } u \in \Sigma_{\mathcal{J}}\}$. It is not hard to see that this $\psi_{\mathfrak{X}}$ enjoys the properties (i), (ii), and (iii) of the measures of noncompactness. With this choice of $\psi_{\mathfrak{X}}$ we prove the following fact.

Lemma 1. *The map $\mathfrak{S}$ is $\psi_{\mathfrak{X}}$-condensing.*

Proof. Let $\mathfrak{C}$ be a closed, bounded subset of $\mathfrak{X}$. Its projection on X, $C = \mathrm{pr}_X(\mathfrak{C})$, is closed and bounded in X. Pick an $\epsilon > 0$. By Assumption 3, there is $\delta > 0$ such that $\sup\limits_{x \in C} d_X(S_i(x), S_j(x)) \le \epsilon$, and hence $d_H(S_i(C), S_j(C)) \le \epsilon$, provided $d_{\mathcal{J}}(i, j) \le \delta$. Let $\mathcal{I}_\delta = \{i_1, \dots, i_R\} \subset \mathcal{J}$ be a finite δ-net in $\mathcal{J}$. We have $\psi_{\mathfrak{X}}\left(\mathfrak{S}(\mathfrak{C})\right) = \psi_X\left(\bigcup_{i \in \mathcal{J}} S_i(C)\right)$. Now, since

$$\left|\psi_X\left(\bigcup_{i \in \mathcal{J}} S_i(C)\right) - \psi_X\left(\bigcup_{j \in \mathcal{I}_\delta} S_j(C)\right)\right| \le c(\psi_X)\, d_H\left(\bigcup_{i \in \mathcal{J}} S_i(C),\ \bigcup_{j \in \mathcal{I}_\delta} S_j(C)\right)$$

and $d_H\left(\bigcup\limits_{i \in \mathcal{J}} S_i(C),\ \bigcup\limits_{j \in \mathcal{I}_\delta} S_j(C)\right) \le \epsilon$, we obtain

$$\psi_{\mathfrak{X}}\left(\mathfrak{S}(\mathfrak{C})\right) \le \psi_X\left(\bigcup_{j \in \mathcal{I}_\delta} S_j(C)\right) + c(\psi_X)\,\epsilon = \psi_X(S_i(C)) + c(\psi_X)\,\epsilon\,,$$

for some $i \in \mathcal{I}_\delta$. Hence, $\psi_{\mathfrak{X}}\left(\mathfrak{S}(\mathfrak{C})\right) \le \psi_X(C) + c(\psi_X)\,\epsilon$, and the inequality is strict if C is not relatively compact. Since $\psi_{\mathfrak{X}}(\mathfrak{C}) = \psi_X(C)$ and ϵ was arbitrary, lemma is proved.

Applying Theorem 1 we immediately obtain the following result.

Theorem 2. *Under the assumptions of Section 2, the dynamics with a range of choice $(\mathfrak{X}, \mathfrak{S})$ possesses a global compact attractor, $\mathfrak{M}$, with all the properties described in Theorem 1.*

The proper definition of the "interval" or IFS dynamics uses the map

$$\overline{F} : A \mapsto \overline{\bigcup_{j \in \mathcal{J}} S_j(A)} \tag{2}$$

acting on closed and bounded subsets of X (in particular, singletons $\{x\}$). We denote the corresponding "semidynamical" system $(X, \bar{F})$.

Theorem 3. *Under the assumptions of Section 2, the IFS $(X, \bar{F})$ has a global compact attractor, K. The attractor K is invariant, $\bar{F}(K) = K$, and it is the maximal closed, bounded, invariant set in X. Through every point x_0 in K passes a full trajectory, i.e., there exists a two-sided infinite sequence $\ldots, x_{-1}, x_0, x_1, \ldots$ of points in K such that $x_{n+1} = S_{j_n}(x_n)$ for some $j_n \in \mathcal{J}$.*

Proof. Although the map $\bar{F}$ is multivalued, all the steps in the proof of Theorem 1 can be carried through thanks to our assumptions and Lemma 1. Indeed, because of Assumption 1 there exists a number $m(\mathbf{B}) > 0$ such that $\bar{F}^n(\mathbf{B}) \subset \mathbf{B}$ for all $n \geq m(\mathbf{B})$. Denote

$$\widetilde{\mathbf{B}} = \overline{\bigcup_{n \geq m(\mathbf{B})} \bar{F}^n(\mathbf{B})}\,.$$

Clearly, $\widetilde{\mathbf{B}} \subset \mathbf{B}$ and $\bar{F}(\widetilde{\mathbf{B}}) \subset \widetilde{\mathbf{B}}$. The proof of Lemma 1 shows that the map $\bar{F}$ is ψ_X-condensing. As follows from [7, Lemma 2.3.5], the set

$$K = \bigcap_{n \geq 1} \bar{F}^n(\widetilde{\mathbf{B}})$$

is a non-empty, $\bar{F}$-invariant compact. Because $\mathbf{B}$ absorbs all bounded sets, K is a global attractor. Its minimality is obvious. This concludes the proof.

The attractors $\mathfrak{M}$ and K are intimately related.

Theorem 4. *The global compact attractor, $\mathfrak{M}$, of the system $(\mathfrak{X}, \mathfrak{S})$ has a product structure, $\mathfrak{M} = K \times \Sigma_{\mathcal{J}}$, where K is the attractor of $(X, \bar{F})$.*

This is an analogue of [9, Theorem 2(iii)] and the proof given in [9] extends to the current case of an infinite parameter space.

It is important to consider dynamics of the system with a fixed plan. This means that, assuming $w \in \Sigma_{\mathcal{J}}$ is the plan, the states of the system change according to the order of the symbols in w: if x_0 is the initial state, then the consecutive states are determined recurrently as $x_{n+1} = S_{w(n)}(x_n)$. For this type of dynamics, many notions and techniques used in the standard theory of attractors make sense and work well. Denote by $w[k]$ the finite part (word) $w(0)w(1)\ldots w(k-1)$ of the string $w \in \Sigma_{\mathcal{J}}$, and denote by $S_{w[k]}$ the composition map $S_{w(k-1)} \circ \cdots \circ S_{w(1)} \circ S_{w(0)}$. The ω-limit set of a set $A \subset X$ is defined as

$$\omega(A, w) = \{y \in X \,:\, \lim_{k \to \infty} S_{w[n_k]}(x_k) = y, \text{ for some } x_k \in A \text{ and } n_k \nearrow \infty\}\,.$$

The definition of the global compact attractor does make sense as well. We call such an attractor corresponding to the plan w the *individual attractor* and denote it $\mathcal{A}_w$.

It is not hard to see that for any bounded $B \subset X$, and any $w \in \Sigma$, $\omega(B, w)$ is not empty, compact, and attracts B.

Theorem 5. *Under the assumptions of Section 2, for every plan $w \in \Sigma_{\mathcal{J}}$ there exists the individual attractor $\mathcal{A}_w$. The attractor $\mathcal{A}_w$ is the union of $\omega(B, w)$ for all bounded $B \subset X$.*

Proof. Define

$$\mathcal{A}_w = \bigcap_{n\geq 1} \overline{\bigcup_{k\geq n} S_{w[k]}(\widetilde{\mathbf{B}})},$$

where the set $\widetilde{\mathbf{B}}$ is defined in the proof of Theorem 3. The proof that $\mathcal{A}_w$ is the attractor and the union of all ω-limit sets is standard in the theory of attractors, see the proof of [9, Theorem 12].

Working with a finite $\mathcal{J}$, in [9, Lemma 13], we showed that $\mathcal{A}_w \subset \bigcup_{j\in\mathcal{J}} S_j(\mathcal{A}_w)$. In the case of infinite $\mathcal{J}$ this is no longer true. However, using Assumption 3, it is not hard to show that $\mathcal{A}_w \subset \bar{F}(\mathcal{A}_w)$. Hence, $\bigcup\limits_{w\in\Sigma} \mathcal{A}_w \subset \overline{F}(\bigcup\limits_{w\in\Sigma} \mathcal{A}_w) \subset K$. In some cases, the individual attractors do cover K. We refer to [9, Lemma 16] for the proof of the following result.

Lemma 2. *$K = \bigcup\limits_{w\in\Sigma} \mathcal{A}_w$ in each of the following cases:*

a) The operators $\{S_j\}$ are strict contractions with the contraction factors $0 < \gamma_j \leq \gamma < 1$.

b) The operators $\{S_j\}$ are eventually strict contractions, i.e., there exist a $0 < \gamma < 1$ and an integer $M \geq 1$ such that for any finite word w^* of length $\geq M$ the operator S_{w^*} is a contraction with the factor γ.

c) $S_j^{-1}(K) \supseteq K$ for $j \in \mathcal{J}$.

d) Each operator S_j is invertible on K.

In [9] we give an example of a dynamics with choice in which K is strictly larger than the union of individual attractors. We say that there is a Gestalt effect in such a case. In our example, $\mathcal{J} = \{0, 1\}$ and $X = \Sigma_{\mathcal{J}}$. The operators S_0 and S_1 are defined on the strings $v = v(0)v(1)v(2)... \in \Sigma_{\mathcal{J}}$ as follows: S_0 writes the third symbol in front of the string, $S_0(v) = v(2).v$, while S_1 writes in front of the string its second symbol, $S_1(v) = v(1).v$. We explain in [9] that the eventually periodic string $u = 000\overline{100}$ belongs to K but not to the union $\bigcup\limits_{w\in\Sigma} \mathcal{A}_w$. In fact, one may check that u is a positive distance away from $\bigcup\limits_{w\in\Sigma} \mathcal{A}_w$.

Here we conjecture that a Gestalt effect is possible only if the maps S_j depend discontinuously on the parameters j.

Conjecture 1. *Assume that $\mathcal{J}$ is an arcwise connected metric compact. Assume that the maps S_j, $j \in \mathcal{J}$, satisfy Assumptions 1, 2, and 3. Then $K = \bigcup\limits_{w\in\Sigma} \mathcal{A}_w$.*

4 Dynamics with restricted choice

An interesting class of systems arises when the choice of the parameters $j \in \mathcal{J}$ at every time step is not arbitrary but is restricted by some rules. For example, consider an oriented, finite or infinite, connected graph such that each vertex has an outgoing edge. Label every edge by a symbol from $\mathcal{J}$ and consider all infinite paths in the graph. The infinite strings of symbols corresponding to the infinite paths form a shift invariant subset of $\Sigma_{\mathcal{J}}$ – the set of allowed (admissible) plans. The operators S_j acting on the states in the order allowed by those plans generate a graph directed dynamics on X, see, e.g., [16] for examples of such systems. More generally, let Λ be a closed, shift invariant subset of $\Sigma_{\mathcal{J}}$. We associate with Λ a discrete time dynamics on the space $\mathfrak{X}_\Lambda = X \times \Lambda$ generated by the iterations of the map $\mathfrak{S}$ defined in (1). This is what we mean by *dynamics with restricted choice.*

Theorem 6. *With Assumptions 1, 2, and 3, the discrete semidynamical system* $(\mathfrak{X}_\Lambda, \mathfrak{S})$ *possesses a global compact attractor* $\mathfrak{M}_\Lambda$.

(1) The attractor $\mathfrak{M}_\Lambda$ *is the maximal invariant compact subset of* $\mathfrak{X}_\Lambda$ *such that* $\mathfrak{S}(\mathfrak{M}_\Lambda) = \mathfrak{M}_\Lambda$. *Clearly,* $\mathfrak{M}_\Lambda$ *is a subset of the global compact attractor* $\mathfrak{M}$ *of the full system* $(\mathfrak{X}, \mathfrak{S})$.

(2) Through every point $(x(0), w)$ *passes a complete trajectory, i.e., there exists a two-sided infinite sequence of points* $..., x(-1), x(0), x(1), ...$ *and a two-sided infinite sequence* $..., w(-1), w(0), w(1), ...$ *(extending* w *in* Λ*) such that* $S_{w(n)}(x(n)) = x(n+1)$ *for all integers* n.

(3) Let K_Λ *denote the projection of the attractor* $\mathfrak{M}_\Lambda$ *onto the* X *component. The set* K_Λ *is a compact subset of the set* K *of Theorem 3. There exist compact sets* A_j, $j \in \mathcal{J}$, *such that* $K_\Lambda = \bigcup_{j\in\mathcal{J}} A_j$ *and* $K_\Lambda = \bigcup_{j\in\mathcal{J}} A_j = \bigcup_{j\in\mathcal{J}} S_j(A_j)$.

(4) If $\Lambda = \Sigma_{\mathcal{J}'}$, *where* $\mathcal{J}'$ *is a closed subset of* $\mathcal{J}$, *then* $\mathfrak{M}_\Lambda = K_\Lambda \times \Lambda$. *In general,* $\mathfrak{M}_\Lambda$ *is not a product, the slices of* $\mathfrak{M}_\Lambda$ *corresponding to different* $w \in \Lambda$ *may be different.*

The proof is the same as the proof of [9, Theorem 3] thanks to Assumptions 1, 2, and 3, and Lemma 1. For examples of $\mathfrak{M}_\Lambda$ with different slices see [9, Section 2.4].

The subshifts over a finite alphabet have been studied extensively, see [15, 10] and references therein. The subshifts over an infinite, possibly uncountable, alphabet have been studied much, much less.

5 Numerical example

The numerical determination of the attractor K in dynamics with choice is a difficult task especially when the parameter space, $\mathcal{J}$, is infinite. Obtaining a verifiable result is even harder. We plan to address these questions in a separate paper. Here we present a simple example.

Consider the following discrete version of the Ross-Macdonald malaria transmission model:

$$x_{n+1} = x_n + \tau\,(a\,y_n\,(1-x_n) - r\,x_n)\,, \quad y_{n+1} = y_n + \tau\,(b\,x_n\,(1-y_n) - m\,y_n)\,. \quad (3)$$

Here τ is the time step, x_n (y_n) is the portion of infected humans (mosquitoes) at the time $n\,\tau$, the coefficients a and b are proportional to the biting rates, r is the recovery rate in humans, and $1/m$ is the average mosquito life-span. The (positive) parameters a, b, r, and m are hard to measure (or even estimate). They depend on many factors (see [19]) that change unpredictably. For the purpose of illustration, we choose $b = 6$, $r = 3$, $m = 2$, and allow a to vary in the interval $[2, 5]$. The time step is $\tau = 0.05$. As long as $\tau < \min\{1/(a+r)\,,\ 1/(b+m)\}$, the state space is the unit square: $X = \{0 \le x \le 1, 0 \le y \le 1\}$. The attractor K is shown on Figure 1. The union of the attractors corresponding to fixed parameters for all a in $[2,5]$ is shown on Figure 2. Each individual attractor geometrically is a curve (heteroclinic orbit) connecting the origin with the fixed point (x_*, y_*), where $x_* = (ab - rm)/(b\,(a+r))$ and $y_* = (ab - rm)/(a\,(b+m))$. One sees that this union is strictly smaller than K. The pictures depend on the size of the time step τ. For larger τ the boundary of the top of the cap (K minus the union) becomes more rugged. What really happens with that part of the boundary (is there an equation for it?) remains a mystery.

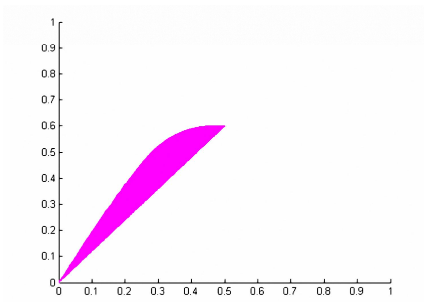

Figure 1: Attractor K.

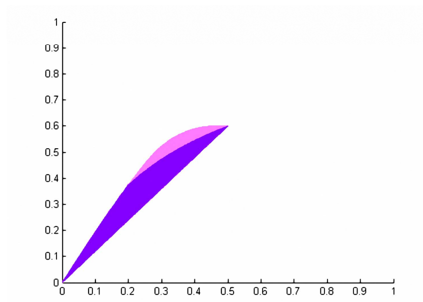

Figure 2: Individual attractors $\mathcal{A}_w$ for $w = jjj\ldots$, all $j \in [2,5]$, superimposed on K.

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