Componentwise Error Estimates for Solutions Obtained by Stationary Iterative Methods*

Shinya Miyajima
Faculty of Engineering, Gifu University, 1-1 Yanagido,
Gifu-shi, Gifu 501-1193, Japan

Abstract

In stationary iterative methods for solving linear systems $Ax = b$, the iteration $x^{(k+1)} = Hx^{(k)} + c$, where H and c are the iteration matrix derived from A and the vector derived from A and b, respectively, is executed for an initial vector $x^{(0)}$. We present a theorem which yields componentwise error estimates for $x^{(k)}$, and clarify the relation between our result and a previous result.

Keywords: stationary iterative methods, linear systems, error estimation
AMS subject classifications: 15A06, 65F10, 65G99

1 Introduction

In this paper, we are concerned with stationary iterative methods for solving linear systems

$$Ax = b, \qquad A \in \mathbb{R}^{n\times n}, \quad x \in \mathbb{R}^n, \quad b \in \mathbb{R}^n. \tag{1}$$

The basic iterative scheme for (1) is

$$Mx^{(k+1)} = Nx^{(k)} + b, \qquad k = 0, 1, \dots \tag{2}$$

where an initial vector $x^{(0)}$ is given, $A = M - N$ and M is nonsingular. The iteration (2) can also be written as

$$x^{(k+1)} = Hx^{(k)} + c, \qquad k = 0, 1, \dots \tag{3}$$

where $H := M^{-1}N$ is the iteration matrix and $c := M^{-1}b$. The iteration (3) converges to the unique solution $x^* = A^{-1}b$ if and only if $\rho(H) < 1$, where $\rho(H)$ denotes the spectral radius of H. The matrix H and the vectors c and x^* satisfy the relation $x^* = Hx^* + c$.

Usually A is decomposed into $A = D+L+U$, where D, L and U are the nonsingular diagonal, strictly lower triangular and strictly upper triangular parts of A, respectively.

*Submitted: October 2, 2011; Revised: March 26, 2012; Accepted: March 26, 2012; Posted: April 6, 2012.

In the Jacobi, Gauss-Seidel and SOR methods (e.g. [1]), M and N are formed as $M = D$ and $N = -L - U$, $M = D + L$ and $N = -U$, and $M = (1/\omega)(D + \omega L)$ and $N = (1/\omega)((1-\omega)D - \omega U)$ for a nonzero real number ω, respectively.

In this paper, we consider error estimation for $x^{(k)}$. Yamamoto [3] established a theorem which yields nonnegative real numbers ε and $\overline{\varepsilon}$ satisfying $\|x^{(k)} - x^*\| \le \varepsilon \le \overline{\varepsilon}$. Namely this theorem gives a *normwise* error estimate for $x^{(k)}$.

The purpose of this paper is to present a theorem which gives *componentwise* error estimates for $x^{(k)}$. This theorem supplies real n-vectors r and $\overline{r}$ satisfying $|x^{(k)} - x^*| \le r \le \overline{r}$, where $|v|$ is the vector of componentwise absolute values of $v \in \mathbb{R}^n$. We prove $\max_{1\le i\le n} r_i \le \varepsilon$ and $\max_{1\le i\le n} \overline{r}_i \le \overline{\varepsilon}$, where v_i is the i-th component of v, if $\|\cdot\|$ is ∞-norm.

2 A Normwise Error Estimate

In this section, we present Theorem 1, which gives a normwise error estimate for $x^{(k)}$ in the iteration (3).

Theorem 1 (Yamamoto [3]) *Let $\|\cdot\|$ be a norm satisfying $\|Fv\| \le \|F\|\|v\|$ for $F \in \mathbb{R}^{n\times n}$ and $v \in \mathbb{R}^n$. In (3), if $\|H\| < 1$, then it follows that*

$$\|x^{(k)} - x^*\| \le \varepsilon \le \overline{\varepsilon},$$

where

$$\varepsilon := \frac{\|x^{(k)} - x^{(k+1)}\|}{1 - \|H\|} \quad \text{and} \quad \overline{\varepsilon} := \frac{\|H\|^k \|x^{(0)} - x^{(1)}\|}{1 - \|H\|}.$$

Proof From $x^{(k+1)} - x^* = (Hx^{(k)} + c) - (Hx^* + c) = H(x^{(k)} - x^*)$, we have

$$\begin{aligned} \|x^{(k)} - x^*\| &\le \|x^{(k)} - x^{(k+1)}\| + \|x^{(k+1)} - x^*\| \\ &\le \|x^{(k)} - x^{(k+1)}\| + \|H\|\|x^{(k)} - x^*\|, \end{aligned}$$

which proves $\|x^{(k)} - x^*\| \le \varepsilon$. Since

$$\begin{aligned} x^{(k)} - x^{(k+1)} &= (Hx^{(k-1)} + c) - (Hx^{(k)} + c) = H(x^{(k-1)} - x^{(k)}) \\ &= \cdots = H^k(x^{(0)} - x^{(1)}), \end{aligned} \tag{4}$$

it follows that

$$\|x^{(k)} - x^{(k+1)}\| = \|H^k(x^{(0)} - x^{(1)})\| \le \|H\|^k \|x^{(0)} - x^{(1)}\|, \tag{5}$$

showing $\varepsilon \le \overline{\varepsilon}$. □

3 Componentwise Error Estimates

In this section, we establish theory yielding componentwise error estimates for $x^{(k)}$ and clarify the relation between the established theory and Theorem 1. Denote the $n \times n$ identity matrix by I. For $M = \{M_{ij}\} \in \mathbb{R}^{m\times n}$, $M^T := \{M_{ji}\}$ and $|M| := \{|M_{ij}|\}$. Let $e := (1, \ldots, 1)^T \in \mathbb{R}^n$. For $v \in \mathbb{R}^n$, v_i denotes the i-th component of v.

We construct Theorem 2, which gives componentwise error estimates for $x^{(k)}$.

Theorem 2 *In (3), if $\|H\|_\infty < 1$, then it follows that*

$$|x^{(k)} - x^*| \le r \le \overline{r},$$

where

$$\begin{aligned} r &:= |x^{(k)} - x^{(k+1)}| + \frac{\|x^{(k)} - x^{(k+1)}\|_\infty}{1 - \|H\|_\infty}|H|e \\ \overline{r} &:= |H|^k|x^{(0)} - x^{(1)}| + \frac{\|H\|_\infty^k\|x^{(0)} - x^{(1)}\|_\infty}{1 - \|H\|_\infty}|H|e. \end{aligned}$$

Proof We have

$$\begin{aligned} x^{(k)} - x^* &= x^{(k)} - x^{(k+1)} + x^{(k+1)} - x^* \\ &= x^{(k)} - x^{(k+1)} + Hx^{(k)} + c - (Hx^* + c) \\ &= x^{(k)} - x^{(k+1)} + H(x^{(k)} - x^*), \end{aligned}$$

so that $(I - H)(x^{(k)} - x^*) = x^{(k)} - x^{(k+1)}$. This and the nonsingularity of $I - H$ give $x^{(k)} - x^* = (I - H)^{-1}(x^{(k)} - x^{(k+1)})$. From this, $\|H\|_\infty < 1$ and Neumann series (*e.g.*, [2]), we obtain

$$\begin{aligned} &|x^{(k)} - x^*| \\ \le\ & |(I - H)^{-1}||x^{(k)} - x^{(k+1)}| \\ =\ & |I + H + H^2 + \cdots||x^{(k)} - x^{(k+1)}| \\ \le\ & (I + |H| + |H|^2 + \cdots)|x^{(k)} - x^{(k+1)}| \\ =\ & |x^{(k)} - x^{(k+1)}| + |H||x^{(k)} - x^{(k+1)}| + |H|(|H||x^{(k)} - x^{(k+1)}|) + \cdots \\ \le\ & |x^{(k)} - x^{(k+1)}| + \|x^{(k)} - x^{(k+1)}\|_\infty|H|e + \||H||x^{(k)} - x^{(k+1)}|\|_\infty|H|e + \cdots \\ =\ & |x^{(k)} - x^{(k+1)}| + (\|x^{(k)} - x^{(k+1)}\|_\infty + \||H||x^{(k)} - x^{(k+1)}|\|_\infty + \cdots)|H|e \\ \le\ & |x^{(k)} - x^{(k+1)}| + (\|x^{(k)} - x^{(k+1)}\|_\infty + \|H\|_\infty\|x^{(k)} - x^{(k+1)}\|_\infty + \cdots)|H|e \\ =\ & |x^{(k)} - x^{(k+1)}| + \|x^{(k)} - x^{(k+1)}\|_\infty(1 + \|H\|_\infty + \|H\|_\infty^2 + \cdots)|H|e \\ =\ & r. \end{aligned}$$

From (4), it follows that

$$|x^{(k)} - x^{(k+1)}| = |H^k(x^{(0)} - x^{(1)})| \le |H|^k|x^{(0)} - x^{(1)}|. \tag{6}$$

The inequalities (5) and (6) prove $r \le \overline{r}$. □

We present Theorem 3, which clarifies the relationship between Theorems 1 and 2, in the case when $\|\cdot\|$ in Theorem 1 is ∞-norm.

Theorem 3 *Let ε and $\overline{\varepsilon}$ be defined as in Theorem 1, and r and $\overline{r}$ be defined as in Theorem 2. If $\|\cdot\|$ in Theorem 1 is ∞-norm and $\|H\|_\infty < 1$, then $\max_{1\le i\le n} r_i \le \varepsilon$ and $\max_{1\le i\le n} \overline{r}_i \le \overline{\varepsilon}$.*

Proof The assumptions imply that

$$\begin{aligned}
\max_{1\le i\le n} r_i &= \|r\|_\infty \\
&\le \|x^{(k)} - x^{(k+1)}\|_\infty + \frac{\|x^{(k)} - x^{(k+1)}\|_\infty}{1-\|H\|_\infty}\||H|e\|_\infty \\
&= \|x^{(k)} - x^{(k+1)}\|_\infty + \frac{\|x^{(k)} - x^{(k+1)}\|_\infty \|H\|_\infty}{1-\|H\|_\infty} \\
&= \varepsilon \qquad (7)
\end{aligned}$$

and

$$\begin{aligned}
\max_{1\le i\le n} \overline{r}_i &= \|\overline{r}\|_\infty \\
&\le \||H|^k|x^{(0)} - x^{(1)}|\|_\infty + \frac{\|H\|_\infty^k\|x^{(0)} - x^{(1)}\|_\infty}{1-\|H\|_\infty}\||H|e\|_\infty \\
&= \||H|^k|x^{(0)} - x^{(1)}|\|_\infty + \frac{\|H\|_\infty^{k+1}\|x^{(0)} - x^{(1)}\|_\infty}{1-\|H\|_\infty} \\
&\le \|H\|_\infty^k\|x^{(0)} - x^{(1)}\|_\infty + \frac{\|H\|_\infty^{k+1}\|x^{(0)} - x^{(1)}\|_\infty}{1-\|H\|_\infty} \\
&= \overline{\varepsilon}. \qquad \Box
\end{aligned}$$

4 Mathematically Rigorous Implementation

Let M, N, D, L and U be as in Section 1, $\|\cdot\|$ and $\overline{\varepsilon}$ be as in Theorem 1, and $\overline{r}$ be as in Theorem 2, and assume $\|\cdot\| = \|\cdot\|_\infty$, $M = D$ and $N = -L-U$ (the case of the Jacobi method). Then $x^{(0)} - x^{(1)} = D^{-1}(Ax^{(0)} - b)$ and $H = -D^{-1}(L+U)$, so rigorous upper bounds for $\|H\|$, $|H|$, $\|x^{(0)} - x^{(1)}\|$ and $|x^{(0)} - x^{(1)}|$ can be computed with $\mathcal{O}(n^2)$ flops via directed rounding, i.e., rigorous upper bounds for $\overline{\varepsilon}$ and $\overline{r}$ can be obtained on a computer with low computational cost. For example, let

$$A = \begin{pmatrix} 1 & -0.5 \\ -0.5 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad k = 50.$$

We computed $\overline{\varepsilon}$ and $\overline{r}$ on a computer with Intel Xeon 2.66GHz Dual CPU, 4.00GB RAM and MATLAB 7.5 with Intel Math Kernel Library and IEEE 754 double precision. Then we obtained $\overline{\varepsilon}$ = 1.8e–15 and $\overline{r}$ = (1.8e–15,8.9e–16)^T.

5 Conclusion

We have constructed Theorem 2 for obtaining an upper bound for $|x^{(k)} - x^*|$. Theorem 3 was presented for clarifying the relationship between Theorem 1 and 2. Our future work will be to develop an algorithm for efficiently and effectively computing upper bounds for $|H|e$ when M and N are not formed as in the Jacobi method.

Acknowledgements

The author wish to thank the referee for helpful comments. This research was partially supported by Grant-in-Aid for Scientific Research (C) (23560066, 2011–2015) from the Ministry of Education, Science, Sports and Culture of Japan.

References

[1] G.H. Golub and C.F. Van Loan. *Matrix Computations, third edition.* The Johns Hopkins University Press, Baltimore and London, 1996.

[2] C.D. Meyer. *Matrix Analysis and Applied Linear Algebra.* SIAM Publications, Philadelphia, 2000.

[3] T. Yamamoto. *Introduction to Numerical Analysis, an enlarged version.* Saiensu-sha, Tokyo, 2003 (in Japanese).