Modal Intervals Revisited
Part 1: A Generalized Interval Natural Extension[*]

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Abstract

Modal interval theory is an extension of classical interval theory which provides richer interpretations (including in particular inner and outer approximations of the ranges of real functions). In spite of its promising potential, modal interval theory is not widely used today because of its original and complicated construction.

The present paper proposes a new formulation of modal interval theory. New extensions of continuous real functions to generalized intervals (intervals whose bounds are not constrained to be ordered) are defined. They are called AE-extensions. These AE-extensions provide the same interpretations as the ones provided by modal interval theory, thus enhancing the interpretation of the classical interval extensions. The construction of AE-extensions strictly follows the model of classical interval theory: starting from a generalization of the definition of the extensions to classical intervals, the minimal AE-extensions of the elementary operations are first built leading to a generalized interval arithmetic. This arithmetic is proved to coincide with the well known Kaucher arithmetic. Then natural AE-extensions are constructed similarly to the classical natural extensions. The natural AE-extensions represent an important simplification of the formulation of the four "theorems of $*$ and $**$ interpretation of a modal rational extension" and "theorems of coercion to $*$ and $**$ interpretability" of modal interval theory. New proofs are provided for the interpretation of these natural AE-extensions that correct the one proposed in the framework of modal intervals.

With a construction similar to classical interval theory, the new formulation of modal interval theory proposed in this paper should facilitate the understanding of the underlying mechanisms, the addition of new items to the theory (e.g. new extensions) and its usage. In particular, a new mean-value extension to generalized intervals will be introduced in the second part of this paper.

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1 Introduction

Classical intervals

The modern interval theory was born in the late 50's aiming reliable computations using finite precision computers (see [38, 36, 22] and [8, 28] for historical and technical introductions). Since its birth, the interval theory has been developed and it proposes today a wide class of useful algorithms independently of the finiteness of computations, although reliable computations are still today an advantage of interval based algorithms (see [16, 12]). The fundamental concept of classical interval theory is the extension of real functions to intervals ([22, 1, 27, 25]). These interval extensions are defined and constructed so as to provide supersets of the ranges of real functions over boxes (i.e. cartesian product of intervals): if g is an interval extension of f and x is box then

$$\{f(x) \mid x \in \boldsymbol{x}\} \subseteq \boldsymbol{g}(\boldsymbol{x}).$$

As a direct application of interval extensions, when dealing with equations $f(x) = 0$ (or equivalently with systems of equations) interval extensions have an intrinsic disproving power: Given an interval extension g of f and a box x, if the interval $\boldsymbol{g}(\boldsymbol{x})$ does not contain 0 then it is proved that the range of f over $\boldsymbol{x}$ does not contain 0 too. As a consequence, the equation $f(x) = 0$ is proved not to have any solution inside $\boldsymbol{x}$. Proving the existence of some solutions to a system of equations can also be done using interval extensions in conjunction with some existence theorems (e.g. Miranda theorem or Brouwer fixed point theorem for $n \times n$ systems of equations): interval extensions are then used in order to check rigorously that the hypothesis of these existence theorems are satisfied (see [23, 24, 15] and [18, 17] for some surveys).

Modal intervals

Modal intervals have been introduced in [6] so as to enhance the interpretation provided by classical interval theory. Modal intervals extend the classical intervals by coupling a quantifier to them: a modal interval is couple $(\boldsymbol{x}, \boldsymbol{q})$ where $\boldsymbol{x}$ is an interval and $\boldsymbol{q}$ a quantifier (see [6, 34, 35]). Real functions are extended to modal intervals taking advantage of the quantifier coupled with the intervals, leading to interpretations richer than the one of the classical interval extensions. In particular, both inner and outer approximations of the range of real functions over boxes are in the scope of modal interval extensions. In contrast to the classical interval extensions, which need to be used in conjunction with some existence theorems to prove the existence of a solution to a system of equations, modal interval extensions have an intrinsic proving power when dealing with systems of equations: indeed, modal interval extensions can build an inner approximation $\boldsymbol{z}$ of the range $\{f(x) \mid x \in \boldsymbol{x}\}$ and if $0 \in \boldsymbol{z}$ then the equation $f(x) = 0$ is proved to have a solution inside $\boldsymbol{x}$.

The enhanced interpretations offered by modal interval theory have promising potential applications (see [37, 2, 29, 30, 31, 9, 10]). However, since its birth, modal interval theory have been used almost only by its creators. This can be explained by some difficulties that are associated with modal interval theory:

(i) The original construction of modal interval theory is complicated and not similar to classical interval theory. For example, two kinds of extensions with different interpretations are present in modal interval theory: the ∗-extensions and the ∗∗-extensions. All interpretation theorems are duplicated to fit both kinds of

modal extensions, although all interpretations obtained with a given kind of extension can also be obtained with the other one. This is misleading and e.g. makes difficult the addition of new components to the theory (for example, some new extensions). On the other hand, modal interval theory emphases the central property that $f^{**}(\boldsymbol{x}) \subseteq f^{*}(\boldsymbol{x})$, but this property between different kinds of modal extensions is quite difficult to interpret. Proposition 9.1 (page 161) shows that this property has a clear interpretation when rephrased using only one type of extension.

(ii) The quantifiers coupled with intervals in modal intervals do not correspond to the quantifiers present inside the interpretations of modal interval extensions. For example, the modal addition

$$(\boldsymbol{x}, \exists) + (\boldsymbol{y}, \forall) = (\boldsymbol{z}, \exists)$$

can be interpreted in two different ways: first, the ∗-interpretation of this modal operation is

$$\big(\forall x \in \boldsymbol{x}\big)\big(\exists z \in \boldsymbol{z}\big)\big(\exists y \in \boldsymbol{y}\big)\big(z = x + y\big).$$

Second, the ∗∗-interpretation of this modal operation is

$$\big(\forall y \in \boldsymbol{y}\big)\big(\forall z \in \boldsymbol{z}\big)\big(\exists x \in \boldsymbol{x}\big)\big(z = x + y\big).$$

In none of the two interpretations the quantifiers coupled with intervals correspond to the quantifiers met in the interpretations.

The aim of the new formulation proposed here is to rephrase the main results of modal interval theory but strictly following the construction of classical interval theory, without introducing pairs of classical intervals, quantifiers, or different kinds of extensions.

The new formulation of modal interval theory

A new formulation of modal interval theory is proposed in the framework of generalized intervals (intervals whose bounds are not constrained to be ordered, initially defined in [26, 13]). This is the main difference with the original formulation of modal interval theory where pairs of classical intervals and quantifiers are introduced while generalized intervals are used as some auxiliary objects that ease computations and proofs.

New extensions to generalized intervals are defined which provide the same enhanced interpretations as the modal interval extensions. They are called AE-extensions because the universal quantifiers (All) are constrained to precede the existential ones (Exist) inside their interpretations[1]. The AE-extensions of continuous real functions are built on the model of classical interval extensions: starting from a generalization of the definition of classical interval extensions, the minimal AE-extensions of the elementary operations (i.e. $+$, $\times$, $-$, $/$, $\exp(x)$, $\sqrt{x}$, etc.) are constructed leading to a generalized interval arithmetic. This arithmetic is proved to coincide with the well known Kaucher arithmetic (an extension of classical interval arithmetic to generalized intervals which has better algebraic properties and which plays a key role in the formal algebraic approach to AE-solution set approximation, [14, 21, 33]). Then, natural AE-extensions are constructed using generalized interval arithmetic in a similar way

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[1]The use of the symbols "AE" in order to insist on the constrained succession of the quantifiers was proposed in [33] in the context of AE-solution sets.

to the classical natural extensions: they consist of evaluating the expression of the function using generalized interval arithmetic. However, in contrast to the classical interval evaluation of an expression, the expression of the function has to be modified before its generalized interval evaluation. A simple algorithm is provided for the modification of the function expression. The natural AE-extensions represent an important simplification of the formulations of the four "theorems of ∗ and ∗∗ interpretation of a modal rational extension" and "theorems of coercion to ∗ and ∗∗ interpretability" initially proposed in modal interval theory. As a result, the new theory overcomes the two main drawbacks pointed out above: AE-extensions correspond to ∗-extensions, and the new theory is developed with no need of ∗∗-extensions.

Some new concepts are also introduced in the new formulation of modal interval theory: in particular, the concept of minimal AE-extensions (i.e. AE-extensions for which no more accurate AE-extension exists) and the concept of the order of convergence of an AE-extension (i.e. some bounds on the distance to some minimal AE-extension) are defined as generalizations of their classical counterparts. They allow to quantify the quality of AE-extensions. It is well known that the classical interval natural extension is minimal if the expression of the function contains only one occurrence of each variable. It is surprising that the natural AE-extension of such function may not be minimal. This was pointed out in [34] but no explanation has been proposed for this fact yet. An explanation is now proposed, see Subsection 12.2. Furthermore, in spite of this lack of minimality, natural AE-extensions are proved to have a linear convergence and to be minimal in the special case of bilinear functions.

Finally, the new theoretical framework proposed here allows providing some new proofs for the new formulations of these theorems. These proofs are based on the study of Skolem functions of quantified propositions, and application of the Brouwer fixed point theorem to these Skolem functions, see Example 11.3 page 169. This new system of proofs was necessary as some gaps have been found in the proofs of some central theorems of modal interval theory (see the counter example in Section B page 177).

Outline of the paper

Generalized intervals and their properties are presented in Section 2. The inclusion between generalized intervals will play an important role. Its interpretation is presented in Section 3. The AE-extensions of real relations are defined in Section 4. They represent a useful language which is used to define the AE-extensions of continuous real functions in Section 5. The minimality of an AE-extension and its order of convergence are defined in Section 6 and Section 7 respectively. The outward rounding to the AE-extensions is presented in Section 8. The special case of real-valued functions is studied in Section 9 where the minimal AE-extension f^* is defined. From the results of the previous section are constructed the minimal AE-extensions of the elementary functions in Section 10 leading to a generalized interval arithmetic. Its relationship with Kaucher arithmetic is investigated. Then, natural AE-extensions are defined in Section 11. The quality and the application scope of natural AE-extensions are finally investigated in Section 12 and 13 respectively.

Contents

Notation

When dealing with sets, the usual set union, set intersection and set difference are respectively denoted by $A \bigcup B$, defined by

$$x \in A \bigcup B \iff x \in A \vee x \in B$$

$A \bigcap B = \{x \in A | x \in B\}$ and $A \backslash B = \{x \in A | x \notin B\}$.

Interval notation follows the one proposed in [19]. Intervals and interval functions will be denoted by boldface letters, e.g. $\boldsymbol{x}$ and $\boldsymbol{f}$. The set of classical intervals (i.e. closed, bounded and nonempty intervals) is denoted by $\mathbb{IR}$. Reals and intervals are identified with real vectors and interval vectors of dimension one, respectively. An interval $\boldsymbol{x} \in \mathbb{IR}^n$ is equivalently considered as a subset of $\mathbb{R}^n$ or as vector of intervals. The interval hull of a nonempty bounded subset E of $\mathbb{R}^n$ is the smallest interval of $\mathbb{IR}^n$ which contains E and is denoted by $\square E$. The lower and upper bounds of an interval (vector) $\boldsymbol{x}$ are denoted respectively by $\inf \boldsymbol{x}$ and $\sup \boldsymbol{x}$. The interval join and meet operations, which are different from the set union and intersection, will be respectively denoted by the symbols $\vee$ and $\wedge$. Given $E \subseteq \mathbb{R}^n$, it will be useful to denote $\{\boldsymbol{x} \in \mathbb{IR}^n | \boldsymbol{x} \subseteq E\}$ by $\boldsymbol{I}E$. The notations of classical intervals will be used for generalized intervals and their related objects, the set of generalized intervals being denoted by $\mathbb{KR}$. The following notation for component numbering of vectors will be used:

Notation. Sets of indices are denoted by calligraphic letters. Let $\mathcal{I} = \{i_1, \ldots, i_n\}$ be an ordered set of indices with $i_k \leq i_{k+1}$. Then, the vector $(x_{i_1}, \ldots, x_{i_n})^T$ is denoted by $x_{\mathcal{I}}$.

This notation is similar to the one proposed in [19]. The involved set of indices will be ordered with the usual lexicographic order. As illustrated by next example, the notation can be applied to reals, intervals or functions.

Example. Consider $\mathcal{I} = \{1, 2, 4\}$ and $\mathcal{I}' = \{(1\,1), (1\,2), 2, (3\,4)\}$. Then,

$$\begin{aligned}
f_{\mathcal{I}} &= (f_1, f_2, f_4)^T \\
x_{\mathcal{I}'} &= (x_{11}, x_{12}, x_2, x_{34})^T \\
x_{\mathcal{I}' \backslash \{2\}} &= (x_{11}, x_{12}, x_{34})^T
\end{aligned}$$

The vector equality is then treated in a natural way: $x_{\mathcal{I}} = y_{\mathcal{I}' \backslash \{2\}}$ stands for $x_1 = y_{11}$, $x_2 = y_{12}$ and $x_4 = y_{34}$.

Intervals of integers are denoted by $[n..m] = \{n, n+1, \ldots, m-1, m\}$ where $n, m \in \mathbb{N}$ with $n \leq m$. The vector $x_{[1..n]} = (x_1, \ldots, x_n)^T$ will be denoted by the usual notation x when no confusion is possible.

The real functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are respectively called real-valued functions and vector-valued functions when emphasis has to be put on this difference.

2 Generalized Intervals

Generalized intervals are now presented. They are the framework of the new formulation of modal interval theory. Furthermore, both the generalized interval inclusion and

generalized interval arithmetic (called Kaucher arithmetic) will play some key roles in the new formulation of modal interval theory.

Classical intervals are closed, bounded and nonempty intervals. They are defined by two bounds: $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$. Generalized intervals extend classical intervals relaxing the constraint that bounds have to be ordered, e.g. $[-1, 1]$ or $[1, -1]$ are generalized intervals. They have been introduced in the late 60's in [26, 13] to improve both the classical interval algebraic structure and the order structure associated with the classical interval inclusion. The set of generalized intervals is denoted by $\mathbb{KR}$ and is decomposed into three subsets:

- The set of proper intervals whose bounds are ordered increasingly. The proper intervals are identified with classical intervals, and therefore to the underlying sets of reals. Therefore, the set of proper intervals is denoted by $\mathbb{IR} = \{[a, b] | a \leq b\}$.

- The set of improper intervals whose bounds are orderer decreasingly. It is denoted by $\overline{\mathbb{IR}} = \{[a, b] | a \geq b\}$.

- The set of degenerate intervals $[a, a]$ with $a \in \mathbb{R}$. These intervals are both proper and improper and are identified with real numbers.

Therefore, from a set of reals $\{x \in \mathbb{R} | a \leq x \leq b\}$, one can build the two generalized intervals $[a, b]$ and $[b, a]$. It will be useful to change one to the other keeping unchanged the underlying set of reals using the following three operations:

- Dual operation: dual $[a, b] = [b, a]$.

- Proper projection: pro $[a, b] = [\min\{a, b\}, \max\{a, b\}] \in \mathbb{IR}$.

- Improper projection: imp $[a, b] = [\max\{a, b\}, \min\{a, b\}] \in \overline{\mathbb{IR}}$.

The classical definitions of midpoint, radius, width and magnitude are extended to generalized intervals keeping the same formula: given a generalized interval $\boldsymbol{x} = [\underline{x}, \overline{x}]$,

$$\text{mid } \boldsymbol{x} = \frac{1}{2}(\overline{x} + \underline{x}) \ \ ; \ \ \text{rad } \boldsymbol{x} = \frac{1}{2}(\overline{x} - \underline{x}) \ \ ; \ \ |\boldsymbol{x}| = \max\{|\underline{x}|, |\overline{x}|\}.$$

The width is defined as wid $\boldsymbol{x} = 2 \,\text{rad } \boldsymbol{x}$. Both the radius and the width are positive for proper intervals and negative for improper intervals. Two generalized intervals are related by $\boldsymbol{x} \leq \boldsymbol{y}$ if and only if sup $\boldsymbol{x} \leq \inf \boldsymbol{y}$. Also, $\boldsymbol{x} < \boldsymbol{y}$ stands for sup $\boldsymbol{x} < \inf \boldsymbol{y}$. When dealing with generalized interval vectors, the same componentwise rules as in classical intervals theory are used. Given a set of indices $\mathcal{K}$ with card $\mathcal{K} = n$ and $\boldsymbol{x}_\mathcal{K} \in \mathbb{KR}^n$, the following functions allow to pick up the indices of the proper and improper components of $\boldsymbol{x}_\mathcal{K}$:

- $\mathcal{P}(\boldsymbol{x}_\mathcal{K}) = \{k \in \mathcal{K} | \boldsymbol{x}_k \in \mathbb{IR}\}$,

- $\mathcal{I}(\boldsymbol{x}_\mathcal{K}) = \{k \in \mathcal{K} | \boldsymbol{x}_k \notin \mathbb{IR}\}$.

Remark 1. Degenerate components are considered as proper intervals by convention. Although the other choice would have been coherent, the convention chosen here is more convenient due to the identification of proper intervals with classical intervals.

Three different structures are attached to the set of generalized intervals: A metric structure, an order structure associated with the generalized interval inclusion and an algebraic structure.

Metric structure

The distance between two generalized intervals $\boldsymbol{x} \in \mathbb{KR}$ and $\boldsymbol{y} \in \mathbb{KR}$ is defined in the following way:

$$\text{dist}(\boldsymbol{x}, \boldsymbol{y}) = \max\{|\underline{x} - \underline{y}|, |\overline{x} - \overline{y}|\}.$$

As shown in [13, 14], $\mathbb{KR}$ then becomes a complete metric space. This metric is extended to $\mathbb{KR}^n$ in the usual way: given $\boldsymbol{x} \in \mathbb{KR}^n$ and $\boldsymbol{y} \in \mathbb{KR}^n$,

$$\text{dist}(\boldsymbol{x}, \boldsymbol{y}) = \max_{k \in [1..n]} \text{dist}(\boldsymbol{x}_k, \boldsymbol{y}_k).$$

A norm is also defined by $||\boldsymbol{x}|| = |\boldsymbol{x}|$ for $\boldsymbol{x} \in \mathbb{KR}$ and by $||\boldsymbol{x}|| = \max_{k \in [1..n]} ||\boldsymbol{x}_k||$ for $\boldsymbol{x} \in \mathbb{KR}^n$.

Order structure associated with the generalized interval inclusion

The generalized intervals are partially ordered by an inclusion which extends the inclusion of classical intervals. Given two generalized intervals $\boldsymbol{x} = [\underline{x}, \overline{x}]$ and $\boldsymbol{y} = [\underline{y}, \overline{y}]$, the inclusion is defined by the same formal expression as the classical interval inclusion:

$$\boldsymbol{x} \subseteq \boldsymbol{y} \iff \underline{y} \le \underline{x} \ \wedge \ \overline{x} \le \overline{y}.$$

Example 2.1. The following inclusions can be checked using the previous definition: $[-1, 1] \subseteq [-2, 2]$, $[2, -2] \subseteq [1, 3]$ and $[2, -2] \subseteq [1, -1]$. The first has a natural interpretation as it coincides with the classical intervals inclusion. The last two will be interpreted in Section 3.

This inclusion is related to the dual operation in the following way:

$$\boldsymbol{x} \subseteq \boldsymbol{y} \iff (\text{dual } \boldsymbol{x}) \supseteq (\text{dual } \boldsymbol{y}).$$

The usual least upper bound and greatest lower bound operations are defined from the inclusion, leading to the join and meet operations of generalized intervals: given a bounded set of generalized intervals $E \subseteq \mathbb{KR}$, its join and meet are respectively denoted by $\vee E$ and $\wedge E$ and are respectively defined by

$$(\forall \boldsymbol{z} \in \mathbb{KR})(\boldsymbol{z} \supseteq (\vee E) \iff \forall \boldsymbol{x} \in E, \boldsymbol{z} \supseteq \boldsymbol{x})$$
$$\text{and} \quad (\forall \boldsymbol{z} \in \mathbb{KR})(\boldsymbol{z} \subseteq (\wedge E) \iff \forall \boldsymbol{x} \in E, \boldsymbol{z} \subseteq \boldsymbol{x}).$$

Remark 2. When E contains only two elements $\boldsymbol{x}$ and $\boldsymbol{y}$, the usual notations $\boldsymbol{x} \vee \boldsymbol{y}$ and $\boldsymbol{x} \wedge \boldsymbol{y}$ are used instead of $\vee\{\boldsymbol{x}, \boldsymbol{y}\}$ and $\wedge\{\boldsymbol{x}, \boldsymbol{y}\}$.

These two definitions lead to the following equivalent characterizations of the generalized interval join and meet (which are formally the same as their classical counterparts):

$$(\vee E) = [\inf_{\boldsymbol{x} \in E} (\inf \boldsymbol{x}) , \sup_{\boldsymbol{x} \in E} (\sup \boldsymbol{x})]$$
$$\text{and} \quad (\wedge E) = [\sup_{\boldsymbol{x} \in E} (\inf \boldsymbol{x}) , \inf_{\boldsymbol{x} \in E} (\sup \boldsymbol{x})].$$

Remark 3. In the context of generalized intervals, it becomes important to use two different signs for the set intersection and for the interval meet. For example, $[0, 1] \bigcap [2, 3] = \emptyset$ whereas $[0, 1] \wedge [2, 3] = [2, 1]$.

Due to the relationship between the inclusion and the operation dual , we have

$$\vee E = \text{dual} \wedge \{\text{dual } \boldsymbol{x} | \boldsymbol{x} \in E\} \quad \text{and} \quad \wedge E = \text{dual} \vee \{\text{dual } \boldsymbol{x} | \boldsymbol{x} \in E\}$$

That is e.g. $\boldsymbol{x} \vee \boldsymbol{y} = \text{dual} \, ((\text{dual } \boldsymbol{x}) \wedge (\text{dual } \boldsymbol{y}))$ in the case where E contains two generalized intervals.

Algebraic structure

In Kaucher[14], continuous real functions are extended to generalized intervals. Given a continuous real function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, its extension to generalized intervals is denoted by $f^{\mathbb{KR}} : \mathbb{KR}^n \longrightarrow \mathbb{KR}$ and is defined in the following way:

$$f^{\mathbb{KR}}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \bigvee\nolimits_{x_1 \in \text{pro } \boldsymbol{x}_1}^{\boldsymbol{x}_1} \cdots \bigvee\nolimits_{x_n \in \text{pro } \boldsymbol{x}_n}^{\boldsymbol{x}_n} f(x)$$

where

$$\bigvee\nolimits^{\boldsymbol{x}} = \begin{cases} \vee & \text{if } \boldsymbol{x} \in \mathbb{IR} \\ \wedge & \text{otherwise} \end{cases}$$

In the case where all the interval arguments are proper, only join operations are involved in the computation and therefore

$$f^{\mathbb{KR}}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \text{range} \, (f, \boldsymbol{x}_1, ..., \boldsymbol{x}_n) \, .$$

In order to differentiate these extensions from the one which will be defined later in this paper, they will be called the $\mathbb{KR}$-extensions of real functions. In the special cases of real arithmetic operations, i.e. $f(x, y) = x \circ y$ with $\circ \in \{+, -, \times, \div\}$, $\mathbb{KR}$-extensions lead to so-called Kaucher arithmetic: these operations between generalized intervals are defined by $\boldsymbol{x} \circ \boldsymbol{y} = f^{\mathbb{KR}}(\boldsymbol{x}, \boldsymbol{y})$. Due to the monotonicity properties of these latter real operations, some simple formula can be derived from the original definition of $\mathbb{KR}$-extensions, leading to the following expressions:

- $\boldsymbol{x} + \boldsymbol{y} = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$.
- $\boldsymbol{x} - \boldsymbol{y} = [\underline{x} - \overline{y}, \overline{x} - \underline{y}] = \boldsymbol{x} + (-\boldsymbol{y})$ where $-\boldsymbol{y} = [-\overline{y}, -\underline{y}]$.
- Kaucher multiplication $\boldsymbol{x} \times \boldsymbol{y}$ is described in Table 1.
- Kaucher division $\boldsymbol{x}/\boldsymbol{y}$ of generalized intervals is defined for generalized intervals $\boldsymbol{x}$ and $\boldsymbol{y}$ such that $0 \notin (\text{pro } \boldsymbol{y})$ by $\boldsymbol{x}/\boldsymbol{y} = \boldsymbol{x} \times [1/\overline{y}, 1/\underline{y}]$.

When restricted to proper intervals, these operations coincide with classical interval arithmetic. The Kaucher arithmetic operations satisfy

$$(\text{dual } \boldsymbol{x}) \circ (\text{dual } \boldsymbol{y}) = \text{dual} \, (\boldsymbol{x} \circ \boldsymbol{y}).$$

We say they commute with the dual operator. This property shows in particular that operations between improper intervals can be computed using classical interval arithmetic, e.g.

$$[2, 1] + [4, 3] = \text{dual} \, ([1, 2] + [3, 4]) = [6, 4].$$

When proper and improper interval are involved, e.g. $[1, 2] + [4, 3] = [5, 5]$, the introduction of classical inner operations is needed to forecast the result of the generalized interval operations using classical ones (see [5]). Kaucher arithmetic has better algebraic properties than classical interval arithmetic: Kaucher addition is a group. The opposite of a generalized interval $\boldsymbol{x}$ is $- \text{dual } \boldsymbol{x}$, i.e.

$$\boldsymbol{x} + (- \text{dual } \boldsymbol{x}) = 0.$$

Table 1: Kaucher multiplication

$\boldsymbol{x} \times \boldsymbol{y}$	$\boldsymbol{y} \in \mathcal{P}$	$\boldsymbol{y} \in \mathcal{Z}$	$\boldsymbol{y} \in -\mathcal{P}$	$\boldsymbol{y} \in \text{dual } \mathcal{Z}$
$\boldsymbol{x} \in \mathcal{P}$	$[\underline{x}\,\underline{y}, \overline{x}\,\overline{y}]$	$[\overline{x}\,\underline{y}, \overline{x}\,\overline{y}]$	$[\overline{x}\,\underline{y}, \underline{x}\,\overline{y}]$	$[\underline{x}\,\underline{y}, \underline{x}\,\overline{y}]$
$\boldsymbol{x} \in \mathcal{Z}$	$[\underline{x}\,\overline{y}, \overline{x}\,\overline{y}]$	$[\min\{\underline{x}\,\overline{y}, \overline{x}\,\underline{y}\},\ \max\{\underline{x}\,\underline{y}, \overline{x}\,\overline{y}\}]$	$[\overline{x}\,\underline{y}, \underline{x}\,\underline{y}]$	0
$\boldsymbol{x} \in -\mathcal{P}$	$[\underline{x}\,\overline{y}, \overline{x}\,\underline{y}]$	$[\underline{x}\,\overline{y}, \underline{x}\,\underline{y}]$	$[\overline{x}\,\overline{y}, \underline{x}\,\underline{y}]$	$[\overline{x}\,\overline{y}, \overline{x}\,\underline{y}]$
$\boldsymbol{x} \in \text{dual } \mathcal{Z}$	$[\underline{x}\,\underline{y}, \overline{x}\,\underline{y}]$	0	$[\overline{x}\,\overline{y}, \underline{x}\,\overline{y}]$	$[\max\{\underline{x}\,\underline{y}, \overline{x}\,\overline{y}\},\ \min\{\underline{x}\,\overline{y}, \overline{x}\,\underline{y}\}]$

where $\mathcal{P} = \{\boldsymbol{x} \in \mathbb{KR} | 0 \leq \underline{x} \wedge 0 \leq \overline{x}\}$, $-\mathcal{P} = \{\boldsymbol{x} \in \mathbb{KR} | 0 \geq \underline{x} \wedge 0 \geq \overline{x}\}$, $\mathcal{Z} = \{\boldsymbol{x} \in \mathbb{KR} | \underline{x} \leq 0 \leq \overline{x}\}$ and dual $\mathcal{Z} = \{\boldsymbol{x} \in \mathbb{KR} | \underline{x} \geq 0 \geq \overline{x}\}$.

Kaucher multiplication restricted to generalized intervals whose bounds are non-null and share the same sign is also a group. The inverse of a such a generalized interval $\boldsymbol{x}$ is $1/(\text{dual } \boldsymbol{x})$, i.e.

$$\boldsymbol{x} \times (1/\text{dual } \boldsymbol{x}) = 1.$$

Finally, the $\mathbb{KR}$-extensions of exp and ln are easily obtained:

$$\exp \boldsymbol{x} = [\exp(\inf \boldsymbol{x}), \exp(\sup \boldsymbol{x})]$$
$$\ln \boldsymbol{x} = [\ln(\inf \boldsymbol{x}), \ln(\sup \boldsymbol{x})] \quad \text{for} \quad \boldsymbol{x} > 0.$$

Furthermore, it is easy to check that for any $\boldsymbol{x} \in \mathbb{KR}$ and $\boldsymbol{y} \in \mathbb{KR}$

$$\exp(\boldsymbol{x} + \boldsymbol{y}) = (\exp \boldsymbol{x})(\exp \boldsymbol{y}) \quad \text{and} \quad \ln \exp \boldsymbol{x} = \boldsymbol{x}.$$

If furthermore $\boldsymbol{x} > 0$ and $\boldsymbol{y} > 0$ then

$$\ln(\boldsymbol{xy}) = \ln \boldsymbol{y} + \ln \boldsymbol{y}.$$

Thus, the reciprocal bijections ln and exp change the additive group of generalized intervals into the multiplicative group of generalized intervals, as in the case of real numbers.

Links between the different structures

The metric and order structures allow the introduction of continuity and monotonicity. The continuity, which will be useful for the coming developments, is local Lipschitz continuity. It is defined in the following way:

Definition 2.1. A function $\boldsymbol{f} : \mathbb{KR}^n \longrightarrow \mathbb{KR}^m$ is locally Lipschitz continuous if and only if for all $\boldsymbol{x}^{\text{ref}} \in \mathbb{IR}^n$, there exists a real $\gamma > 0$ such that for any generalized intervals $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{K}\boldsymbol{x}^{\text{ref}}$,

$$\text{dist}\big(\boldsymbol{f}(\boldsymbol{x}), \boldsymbol{f}(\boldsymbol{y})\big) \leq \gamma \,\text{dist}(\boldsymbol{x}, \boldsymbol{y})$$

Remark 4. This definition is naturally specialized to functions $\boldsymbol{f} : \mathbb{IR}^n \longrightarrow \mathbb{IR}^m$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ considering respectively all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{I}\boldsymbol{x}^{\text{ref}}$ and all $x, y \in \boldsymbol{x}^{\text{ref}}$.

It is shown in Kaucher[14] that $\mathbb{KR}$-extensions of locally Lipschitz continuous real functions are locally Lipschitz continuous (this holds in particular for Kaucher arithmetic). Furthermore, Kaucher arithmetic operations are inclusion monotone, i.e.

$$\boldsymbol{x} \subseteq \boldsymbol{x}' \Longrightarrow \boldsymbol{x} \circ \boldsymbol{y} \subseteq \boldsymbol{x}' \circ \boldsymbol{y}$$

In the special cases of addition and multiplication by intervals whose proper projection does not contain zero, the equivalence holds in place of the implication. Finally, the metric and norm are related by $\text{dist}(\boldsymbol{x}, \boldsymbol{y}) = ||\boldsymbol{x} - \text{dual } \boldsymbol{y}||$.

3 Interpretation of Generalized Interval Inclusions

When restricted to proper intervals, the generalized interval inclusion coincides with the classical interval inclusion and therefore has a natural interpretation in terms of sets of reals:

$$\boldsymbol{x} \subseteq \boldsymbol{y} \iff \{x \in \mathbb{R} | x \in \boldsymbol{x}\} \subseteq \{y \in \mathbb{R} | y \in \boldsymbol{y}\}$$

An interpretation in terms of sets of reals is now provided for the other cases of inclusion between generalized intervals. The next lemma first proves that the inclusion between a proper interval and an improper interval is related to the intersection between the related sets of reals (this property was already pointed out e.g. in [32]).

Lemma 3.1. *Let* $\boldsymbol{x} \in \mathbb{IR}$ *and* $\boldsymbol{y} \in \mathbb{IR}$. *Then,*

$$\boldsymbol{x} \bigcap \boldsymbol{y} \neq \emptyset \iff (\text{dual } \boldsymbol{x}) \subseteq \boldsymbol{y}$$

Proof. Clearly, the proper intervals $\boldsymbol{x}$ and $\boldsymbol{y}$ have a non-null intersection if and only if $\neg (\overline{y} < \underline{x} \vee \overline{x} < \underline{y})$. Distributing the negation, one obtains the equivalent condition $\overline{y} \geq \underline{x} \wedge \overline{x} \geq \underline{y}$. It remains to notice that $(\text{dual } \boldsymbol{x}) = [\overline{x}, \underline{x}]$ so using the expression of inclusion we eventually have $(\text{dual } \boldsymbol{x}) \subseteq \boldsymbol{y} \iff \overline{y} \geq \underline{x} \wedge \overline{x} \geq \underline{y}$. $\square$

Example 3.1. Consider the intervals $\boldsymbol{x} = [-1, 1]$ and $\boldsymbol{y} = [0, 2]$. On one hand $\boldsymbol{x} \bigcap \boldsymbol{y} \neq \emptyset$. On the other hand $(\text{dual } \boldsymbol{x}) \subseteq \boldsymbol{y}$, i.e. $[1, -1] \subseteq [0, 2]$, can be checked using the rules of generalized interval inclusion ($0 \leq 1$ and $-1 \leq 2$).

The following proposition allows interpretation of all the the cases of inclusion between generalized intervals in terms of proper intervals, i.e. in terms of sets of real numbers:

Proposition 3.1. *Let* $\boldsymbol{x} \in \mathbb{KR}$ *and* $\boldsymbol{y} \in \mathbb{KR}$. *Then,* $\boldsymbol{x} \subseteq \boldsymbol{y}$ *is equivalent to the disjunction of the four following conditions:*

1. $\boldsymbol{x} \in \mathbb{IR} \wedge \boldsymbol{y} \in \mathbb{IR} \wedge \boldsymbol{x} \subseteq \boldsymbol{y}$

2. $\boldsymbol{x} \in \overline{\mathbb{IR}} \wedge \boldsymbol{y} \in \mathbb{IR} \wedge (\text{pro } \boldsymbol{x}) \bigcap \boldsymbol{y} \neq \emptyset$

3. $\boldsymbol{x} \in \overline{\mathbb{IR}} \wedge \boldsymbol{y} \in \overline{\mathbb{IR}} \wedge (\text{pro } \boldsymbol{x}) \supseteq (\text{pro } \boldsymbol{y})$

4. $\boldsymbol{x} \in \mathbb{IR} \wedge \boldsymbol{y} \in \overline{\mathbb{IR}} \wedge \boldsymbol{x} = \boldsymbol{y} = [x, x]$.

Proof. The inclusion $\boldsymbol{x} \subseteq \boldsymbol{y}$ is equivalent to

$$(\boldsymbol{x} \in \mathbb{IR} \wedge \boldsymbol{y} \in \mathbb{IR} \wedge \boldsymbol{x} \subseteq \boldsymbol{y}) \ \vee \ (\boldsymbol{x} \in \overline{\mathbb{IR}} \ \wedge \ \boldsymbol{y} \in \mathbb{IR} \wedge \boldsymbol{x} \subseteq \boldsymbol{y})$$
$$\vee \ (\boldsymbol{x} \in \overline{\mathbb{IR}} \ \wedge \ \boldsymbol{y} \in \overline{\mathbb{IR}} \wedge \boldsymbol{x} \subseteq \boldsymbol{y}) \ \vee \ (\boldsymbol{x} \in \mathbb{IR} \ \wedge \ \boldsymbol{y} \in \overline{\mathbb{IR}} \wedge \boldsymbol{x} \subseteq \boldsymbol{y})$$

because a generalized interval is either proper or improper. Therefore the four cases can be proved independently. **(1)** This case corresponds to the classical interval inclusion. **(2)** One can apply Lemma 3.1 to obtain $(\text{pro } \boldsymbol{x}) \bigcap \boldsymbol{y} \neq \emptyset \iff (\text{imp } \boldsymbol{x}) \subseteq \boldsymbol{y}$. Finally, since $\boldsymbol{x}$ is improper, $(\text{imp } \boldsymbol{x}) = \boldsymbol{x}$ and therefore $(\text{pro } \boldsymbol{x}) \bigcap \boldsymbol{y} \neq \emptyset \iff \boldsymbol{x} \subseteq \boldsymbol{y}$. **(3)** First $\boldsymbol{x} \subseteq \boldsymbol{y} \iff (\text{dual } \boldsymbol{x}) \supseteq (\text{dual } \boldsymbol{y})$ and remark that, $\boldsymbol{x}$ and $\boldsymbol{y}$ being improper, $(\text{dual } \boldsymbol{x}) = (\text{pro } \boldsymbol{x})$ and $(\text{dual } \boldsymbol{y}) = (\text{pro } \boldsymbol{y})$ and therefore $(\text{pro } \boldsymbol{y}) \subseteq (\text{pro } \boldsymbol{x}) \iff \boldsymbol{x} \subseteq \boldsymbol{y}$. **(4)** This last case is less interesting. This is a direct consequence of the definition of inclusion. $\qquad \square$

An example is now proposed for first three cases of the previous proposition.

Example 3.2.

1. $[-1, 1] \subseteq [-2, 2]$ is interpreted like the classical set inclusion;

2. $[2, -2] \subseteq [1, 3]$ is interpreted as $[-2, 2] \bigcap [1, 3] \neq \emptyset$;

3. $[2, -2] \subseteq [1, -1]$ is interpreted as $[-1, 1] \subseteq [-2, 2]$.

The next corollary of cases (2) and (3) of Proposition 3.1 will be useful.

Corollary 3.1. *Let $\boldsymbol{x} \in \overline{\mathbb{IR}}$ and $\boldsymbol{y} \in \mathbb{KR}$. Then,*

$$\boldsymbol{x} \subseteq \boldsymbol{y} \iff \left(q^{(\boldsymbol{y})} t \in \text{pro } \boldsymbol{y}\right)\left(t \in \text{pro } \boldsymbol{x}\right)$$

where $q^{(\boldsymbol{y})} = \exists$ if $\boldsymbol{y} \in \mathbb{IR}$ and $q^{(\boldsymbol{y})} = \forall$ otherwise.

Proof. First suppose that $\boldsymbol{y} \in \mathbb{IR}$. Then, thanks to Proposition 3.1, $\boldsymbol{x} \subseteq \boldsymbol{y} \iff (\text{pro } \boldsymbol{x}) \bigcap \boldsymbol{y} \neq \emptyset$ which is eventually equivalent to $\left(\exists t \in \boldsymbol{y}\right)\left(t \in \text{pro } \boldsymbol{x}\right)$. Now suppose that $\boldsymbol{y} \in \overline{\mathbb{IR}}$. Then thanks to Proposition 3.1, $\boldsymbol{x} \subseteq \boldsymbol{y} \iff (\text{pro } \boldsymbol{x}) \supseteq (\text{pro } \boldsymbol{y})$ which is eventually equivalent to $\left(\forall t \in \text{pro } \boldsymbol{y}\right)\left(t \in \text{pro } \boldsymbol{x}\right)$. $\qquad \square$

This section is ended with an informal presentation of the interpretation of the join and meet operations between generalized intervals. These interpretations are not detailed here because they will not be used in the sequel. They are direct consequences of Proposition 3.1 and of the basic definitions of least upper bound and greatest lower bound.

Example 3.3. The following four cases summarize the possible interpretations of the generalized interval meet operation:

- $[-1, 1] \wedge [0, 2] = [0, 1]$ so the proper interval $[0, 1]$ is the biggest interval included inside both $[-1, 1]$ and $[0, 2]$. This corresponds to the interpretation of the classical interval meet operation.

- $[-1, 1] \wedge [2, 3] = [2, 1]$ so the proper interval $[1, 2]$ is interpreted as the smallest interval which intersects both $[-1, 1]$ and $[2, 3]$.

- $[-1, 1] \wedge [3, 2] = [3, 1]$ so the proper interval $[1, 3]$ is interpreted as the smallest interval which both intersects $[-1, 1]$ and contains $[2, 3]$.

- $[1, -1] \wedge [3, 2] = [3, -1]$ so the proper interval $[-1, 3]$ is interpreted as the smallest interval which contains both $[-1, 1]$ and $[2, 3]$.

The join and meet being related by

$$\boldsymbol{x} \vee \boldsymbol{y} = \boldsymbol{z} \iff (\operatorname{dual} \boldsymbol{x}) \wedge (\operatorname{dual} \boldsymbol{x}) = (\operatorname{dual} \boldsymbol{z})$$

the interpretations of the join operation between generalized intervals are obtained thanks to the interpretations of the meet operation.

4   Quantified Propositions and Generalized Intervals

This section presents the quantified propositions that will be met in the sequel. These quantified propositions will be expressed as extensions of real relations to generalized intervals leading to a convenient language for their manipulation.

4.1   Quantified Propositions in AE-Form

The quantified propositions encountered in the sequel are of the following kind: they are closed (no free variables), in prenex form (the quantifiers occur in front of the proposition) and in AE-form (the universal quantifiers precede the existential ones). Furthermore, the quantifiers act over interval domains. Such quantified propositions can be written in the following way:

$$\big(\forall x_{\mathcal{A}} \in \boldsymbol{x}_{\mathcal{A}}\big)\big(\exists x_{\mathcal{E}} \in \boldsymbol{x}_{\mathcal{E}}\big)\big(\phi(x_{\mathcal{A} \bigcup \mathcal{E}})\big) \tag{1}$$

where $\mathcal{A}$ and $\mathcal{E}$ are disjoint sets of indices such that $\operatorname{card} \mathcal{A} + \operatorname{card} \mathcal{E} = n$ and $\boldsymbol{x}_{\mathcal{A} \bigcup \mathcal{E}} \in \mathbb{IR}^n$ and ϕ is a real relation of $\mathbb{R}^n$.

Remark 5. The quantified proposition (1) is actually a short cut for the exact formulation

$$\big(\forall x_{\mathcal{A}}\big)\big(\exists x_{\mathcal{E}}\big)\big(x_{\mathcal{A}} \in \boldsymbol{x}_{\mathcal{A}} \implies x_{\mathcal{E}} \in \boldsymbol{x}_{\mathcal{E}} \wedge \phi(x)\big),$$

where the domains of x and y are respectively $\mathbb{R}^n$ and $\mathbb{R}^m$ (see e.g. [39]).

It will be convenient to define Skolem functions of quantified propositions like (1). Using an analogy with first order logic Skolem functions (see e.g. [39]), a Skolem function of (1) is a function

$$s_{\mathcal{E}} : \boldsymbol{x}_{\mathcal{A}} \longrightarrow \boldsymbol{x}_{\mathcal{E}} \quad \text{s.t.} \quad x_{\mathcal{E}} = s_{\mathcal{E}}(x_{\mathcal{A}}) \implies \phi(x_{\mathcal{A} \bigcup \mathcal{E}}).$$

Remark 6. It is implied that the previous implication stands for all $x_{\mathcal{A}} \in \boldsymbol{x}_{\mathcal{A}}$.

Obviously, the quantified proposition (1) is true if and only if it has a Skolem function.

Example 4.1. Consider the relation $\phi(x)$ of $\mathbb{R}^2$ defined by $\phi(x) \iff x_1^2 + x_2^2 - 9 = 0$. The quantified proposition $\big(\forall x_1 \in [-1, 1]\big)\big(\exists x_2 \in [2, 4]\big)\big(\phi(x)\big)$ has the following Skolem function: $s_2 : [-1, 1] \longrightarrow [2, 4]$ defined by $s_2(x_1) = \sqrt{9 - x_1^2}$. Indeed $x_2 = s(x_1)$ implies $\phi(x)$. As a consequence, the previous quantified proposition is true.

An important topic in the sequel will be to prove that a quantified proposition has a **continuous** Skolem function. This can be understood as the possibility to choose the existentially quantified variables continuously with respect to the values of the universally quantified variables.

4.2 AE-Extensions of Real Relations

Given a real relation ϕ of $\mathbb{R}^n$, its extension to generalized intervals is now defined. The AE-extensions of real relations are introduced because they represent a useful language for the description of the quantified propositions like (1) and of their properties.

Definition 4.1. Let ϕ be a relation of $\mathbb{R}^n$. The AE-extension of ϕ is denoted by the same symbol ϕ and is defined for $\boldsymbol{x} \in \mathbb{KR}^n$ by

$$\phi(\boldsymbol{x}) \iff \big(\forall x_{\mathcal{I}} \in \text{pro } \boldsymbol{x}_{\mathcal{I}}\big)\big(\exists x_{\mathcal{P}} \in \boldsymbol{x}_{\mathcal{P}}\big)\big(\phi(x)\big)$$

where $\mathcal{P} = \mathcal{P}(\boldsymbol{x})$ and $\mathcal{I} = \mathcal{I}(\boldsymbol{x})$ are respectively the set of indices of proper and improper components of $\boldsymbol{x}$.

Remark 7. The general notation which is used in the definition 4.1 also stands for the cases where $\mathcal{P} = \emptyset$ or $\mathcal{I} = \emptyset$. These cases correspond respectively to $\big(\forall x \in \boldsymbol{x}\big)\big(\phi(x)\big)$ and $\big(\exists x \in \boldsymbol{x}\big)\big(\phi(x)\big)$.

When restricted to proper intervals, the definition proposed here does not exactly coincide with the definition proposed in [4, 3] where an implication is involved in place of the equivalence. The following examples illustrate the interpretations of AE-extensions of real relations.

Example 4.2. Consider the relation $\phi(x)$ of $\mathbb{R}^2$ defined by $\phi(x) \iff x_1^2 + x_2^2 - 9 = 0$ and the generalized interval $\boldsymbol{x} = ([1, -1], [2, 4])^T$. Then $\phi(\boldsymbol{x})$ is equivalent to the quantified proposition $\big(\forall x_1 \in [-1, 1]\big)\big(\exists x_2 \in [2, 4]\big)\big(\phi(x)\big)$ and is true as shown in Example 4.1.

The next example illustrates the way AE-extensions of real relations will be related to interval extensions of real functions in the next section.

Example 4.3. Consider a continuous function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and two intervals $\boldsymbol{x} \in \mathbb{KR}^n$ and $\boldsymbol{z} \in \mathbb{IR}^m$. Denote by $\phi(x, z)$ the real relation of $\mathbb{R}^{n+m}$ defined by $\phi(x, z) \iff f(x) = z$. Now, if $\boldsymbol{x}$ is proper then the evaluation $\phi(\text{dual } \boldsymbol{x}, \boldsymbol{z})$ is equivalent to the following quantified proposition: $\big(\forall x \in \boldsymbol{x}\big)\big(\exists z \in \boldsymbol{z}\big)\big(f(x) = z\big)$. Indeed, dual $\boldsymbol{x}$ is improper so x is universally quantified and $\boldsymbol{z}$ is proper so z is existentially quantified. Therefore, $\phi(\text{dual } \boldsymbol{x}, \boldsymbol{z})$ is equivalent to range $(f, \boldsymbol{x}) \subseteq \boldsymbol{z}$. Now, if $\boldsymbol{x}$ is improper then dual $\boldsymbol{x}$ is proper and $\phi(\text{dual } \boldsymbol{x}, \boldsymbol{z})$ is equivalent to the following quantified proposition: $\big(\exists x \in \text{pro } \boldsymbol{x}\big)\big(\exists z \in \boldsymbol{z}\big)\big(f(x) = z\big)$, i.e. range $(f, \boldsymbol{x}) \bigcap \boldsymbol{z} \neq \emptyset$.

The central role played by the generalized interval inclusion in the coming developments is due to its strong relationship with the AE-extensions of real relations. This important relationship is described by the following proposition:

Proposition 4.1. *Let ϕ be a relation of $\mathbb{R}^n$, $\boldsymbol{x} \in \mathbb{KR}^n$ and $\boldsymbol{y} \in \mathbb{KR}^n$. Then*

$$\boldsymbol{x} \subseteq \boldsymbol{y} \wedge \phi(\boldsymbol{x}) \implies \phi(\boldsymbol{y}).$$

Proof. Define the following sets of indices:

$$\mathcal{P} = \mathcal{P}(\boldsymbol{x}) \bigcap \mathcal{P}(\boldsymbol{y}) \ ; \ \mathcal{I} = \mathcal{I}(\boldsymbol{x}) \bigcap \mathcal{I}(\boldsymbol{y}) \ ; \ \mathcal{K} = [1..n] \backslash (\mathcal{P} \bigcup \mathcal{I}).$$

We suppose that the three previously defined set of indices are not empty (the other cases are similar and simpler). Thanks to Proposition 3.1, we know that the inclusion $\boldsymbol{x} \subseteq \boldsymbol{y}$ implies

- $k \in \mathcal{P}$ implies $\boldsymbol{x}_k \subseteq \boldsymbol{y}_k$ and therefore $\boldsymbol{x}_{\mathcal{P}} \subseteq \boldsymbol{y}_{\mathcal{P}}$

- $k \in \mathcal{I}$ implies pro $\boldsymbol{y}_k \subseteq$ pro $\boldsymbol{x}_k$ and therefore pro $\boldsymbol{y}_{\mathcal{I}} \subseteq$ pro $\boldsymbol{x}_{\mathcal{I}}$

- $k \in \mathcal{K}$ implies $\boldsymbol{x}_k \in \overline{\mathbb{IR}}$, $\boldsymbol{y}_k \in \mathbb{IR}$ and (pro $\boldsymbol{x}_k) \bigcap \boldsymbol{y}_k \neq \emptyset$ and therefore (pro $\boldsymbol{x}_{\mathcal{K}}) \bigcap \boldsymbol{y}_{\mathcal{K}} \neq \emptyset$

$\phi(\boldsymbol{x})$ is true if and only if $(\forall x_{\mathcal{I}} \in \text{pro } \boldsymbol{x}_{\mathcal{I}})(\forall x_{\mathcal{K}} \in \text{pro } \boldsymbol{x}_{\mathcal{K}})(\exists x_{\mathcal{P}} \in \boldsymbol{x}_{\mathcal{P}})(\phi(x))$ and $\phi(\boldsymbol{y})$ is true if and only if $(\forall x_{\mathcal{I}} \in \text{pro } \boldsymbol{y}_{\mathcal{I}})(\exists x_{\mathcal{K}} \in \boldsymbol{y}_{\mathcal{K}})(\exists x_{\mathcal{P}} \in \boldsymbol{y}_{\mathcal{P}})(\phi(x))$. We just have to prove that the former implies the latter. This is obviously true because $\boldsymbol{x}_{\mathcal{P}} \subseteq \boldsymbol{y}_{\mathcal{P}}$, pro $\boldsymbol{y}_{\mathcal{I}} \subseteq$ pro $\boldsymbol{x}_{\mathcal{I}}$ and (pro $\boldsymbol{x}_{\mathcal{K}}) \bigcap \boldsymbol{y}_{\mathcal{K}} \neq \emptyset$. $\qquad \square$

Example 4.4. Consider the relation $\phi(x)$ of $\mathbb{R}^2$ defined by $\phi(x) \iff x_1^2 + x_2^2 - 9 = 0$ and $\boldsymbol{x} = ([1, -1], [2, 4])^T$ and $\boldsymbol{y} = ([0.9, -0.9], [1.9, 4.1])^T$. Then both $\boldsymbol{x} \subseteq \boldsymbol{y}$ and $\phi(\boldsymbol{x})$ are true, so $\phi(\boldsymbol{y})$ is true by Proposition 4.1. Indeed, the quantified proposition $(\forall x_1 \in [-1, 1])(\exists x_2 \in [2, 4])(\phi(x))$ clearly implies the quantified proposition $(\forall x_1 \in [-0.9, 0.9])(\exists x_2 \in [1.9, 4.1])(\phi(x))$.

Finally, the next two propositions are technical results which will be useful in the sequel.

Proposition 4.2. *Let ϕ be a relation of $\mathbb{R}^n$ and $\boldsymbol{x} \in \mathbb{KR}^n$ be a generalized interval. When $\square\{x \in (\text{pro } \boldsymbol{x})|\phi(x)\}$ is defined (i.e. when $\{x \in (\text{pro } \boldsymbol{x})|\phi(x)\}$ is nonempty), denote this interval by $\boldsymbol{a}$. Then the three following implications are true:*

(i) $\phi(\boldsymbol{x}) \Longrightarrow \{x \in (\text{pro } \boldsymbol{x})|\phi(x)\} \neq \emptyset$

(ii) $\phi(\boldsymbol{x}) \Longrightarrow (\text{imp } \boldsymbol{a}) \subseteq \boldsymbol{x}$

(iii) $\phi(\boldsymbol{x}) \Longrightarrow \phi(\boldsymbol{x} \wedge \boldsymbol{a})$.

Proof. Define the set of indices $\mathcal{P} = \mathcal{P}(\boldsymbol{x})$ and $\mathcal{I} = \mathcal{I}(\boldsymbol{x})$. Suppose that both P and I are not empty, the other cases being similar and simpler.

(i) As $\phi(\boldsymbol{x})$ is true the following quantified proposition holds:

$$(\forall x_{\mathcal{I}} \in \text{pro } \boldsymbol{x}_{\mathcal{I}})(\exists x_{\mathcal{P}} \in \boldsymbol{x}_{\mathcal{P}})(\phi(x)). \tag{2}$$

Choose $\tilde{x}_{\mathcal{I}} \in \text{pro } \boldsymbol{x}_{\mathcal{I}}$. By the previous quantified proposition, there exists $\tilde{x}_{\mathcal{P}} \in \boldsymbol{x}_{\mathcal{P}}$ such that $\phi(\tilde{x})$ holds. Therefore $\tilde{x} \in \{x \in (\text{pro } \boldsymbol{x})|\phi(x)\}$ which concludes the proofs of the first assertion.

(ii) On one hand, the definition of $\boldsymbol{a}$ obviously implies $\boldsymbol{a} \subseteq \text{pro } \boldsymbol{x}$ and hence $\boldsymbol{a}_{\mathcal{P}} \subseteq \boldsymbol{x}_{\mathcal{P}}$. Therefore $(\text{imp } \boldsymbol{a}_{\mathcal{P}}) \subseteq \boldsymbol{x}_{\mathcal{P}}$. On the other hand, define $u_{\mathcal{I}} = \inf \boldsymbol{x}_{\mathcal{I}}$ and $v_{\mathcal{I}} = \sup \boldsymbol{x}_{\mathcal{I}}$ so by the quantified proposition (2) there exists $u_{\mathcal{P}} \in \boldsymbol{x}_{\mathcal{P}}$ and $v_{\mathcal{P}} \in \boldsymbol{x}_{\mathcal{P}}$ such that $\phi(u)$ and $\phi(v)$. Therefore, $u \in \boldsymbol{a}$ and $v \in \boldsymbol{a}$ and finally $u \vee v \subseteq \boldsymbol{a}$. Now by construction $(u \vee v)_{\mathcal{I}} = \text{pro } \boldsymbol{x}_{\mathcal{I}}$ so $(\text{pro } \boldsymbol{x}_{\mathcal{I}}) \subseteq \boldsymbol{a}_{\mathcal{I}}$. This implies $(\text{dual } \boldsymbol{a}_{\mathcal{I}}) \subseteq (\text{dual pro } \boldsymbol{x}_{\mathcal{I}})$ and because $\boldsymbol{a}_{\mathcal{I}}$ is proper and $\boldsymbol{x}_{\mathcal{I}}$ is improper $(\text{imp } \boldsymbol{a}_{\mathcal{I}}) \subseteq \boldsymbol{x}_{\mathcal{I}}$.

(iii) Denote $\boldsymbol{x} \wedge \boldsymbol{a}$ by $\boldsymbol{z}$. We have proved previously that $\boldsymbol{a}_{\mathcal{P}} \subseteq \boldsymbol{x}_{\mathcal{P}}$ and $(\text{pro } \boldsymbol{x}_{\mathcal{I}}) \subseteq \boldsymbol{a}_{\mathcal{I}}$ the latter implying $\boldsymbol{x}_{\mathcal{I}} \subseteq \boldsymbol{a}_{\mathcal{I}}$ because $\boldsymbol{x}_{\mathcal{I}}$ is improper. Therefore $\boldsymbol{z}_{\mathcal{P}} = \boldsymbol{a}_{\mathcal{P}}$ and $\boldsymbol{z}_{\mathcal{I}} = \boldsymbol{x}_{\mathcal{I}}$. Now as $\phi(\boldsymbol{x})$ is true the quantified proposition corresponding to its interpretation (i.e. (2)) has a Skolem function, i.e. there exists a function $s_{\mathcal{P}} : \text{pro } \boldsymbol{x}_{\mathcal{I}} \longrightarrow \boldsymbol{x}_{\mathcal{P}}$ such that $x_{\mathcal{P}} = s_{\mathcal{P}}(x_{\mathcal{I}}) \Longrightarrow \phi(x)$. Now, by construction of $\boldsymbol{a}$, for any $x_{\mathcal{I}} \in \boldsymbol{x}_{\mathcal{I}}$, $\phi(x)$ implies $x_{\mathcal{P}} \in \boldsymbol{a}_{\mathcal{P}}$. Therefore the actual range of s is a subset of $\boldsymbol{a}_{\mathcal{P}}$. As a consequence, the quantified proposition that corresponds to the interpretation of $\phi(\boldsymbol{z})$ has a Skolem function and $\phi(\boldsymbol{z})$ is finally true. $\qquad \square$

The next proposition needs the following lemma which is a direct consequence of the fact that the distance between proper intervals corresponds to the Hausdorff distance between the corresponding sets of reals:

$$\text{dist}(\boldsymbol{x}, \boldsymbol{y}) = \max\{ \max_{y \in \boldsymbol{y}} \min_{x \in \boldsymbol{x}} \text{dist}(x, y) \ , \ \max_{x \in \boldsymbol{x}} \min_{y \in \boldsymbol{y}} \text{dist}(x, y) \ \}$$

where the distance between reals is $\text{dist}(x, y) = \max_{k \in [1..n]} |x_k - y_k|$.

Lemma 4.1. *Let $\boldsymbol{z}^{(k)} \in \mathbb{IR}^p$ be a sequence of intervals which converges to $\boldsymbol{z} \in \mathbb{IR}^p$. Then both following statements are true.*

  (i) Any sequence $(z^{(k)})_{k \in \mathbb{N}}$ which satisfies $z^{(k)} \in \boldsymbol{z}^{(k)}$ has at least one accumulation point and all its accumulation points are in $\boldsymbol{z}$.

  (ii) For any $z \in \boldsymbol{z}$, there exists a convergent sequence $(z^{(k)})_{k \in \mathbb{N}}$ which satisfies $z^{(k)} \in \boldsymbol{z}^{(k)}$ whose limit is z.

Proof. (i) The sequence $\boldsymbol{z}^{(k)}$ being convergent, it is also bounded. Therefore, any sequence $z^{(k)} \in \boldsymbol{z}^{(k)}$ is bounded. This latter hence have at least one accumulation point thanks to the Bolzano-Weierstrass theorem. By the definition of the Hausdorff distance and because $z^{(k)} \in \boldsymbol{z}^{(k)}$, we have whatever is $k \in \mathbb{N}$

$$\text{dist}(\boldsymbol{z}^{(k)}, \boldsymbol{z}) \geq \max_{z \in \boldsymbol{z}^{(k)}} \min_{y \in \boldsymbol{z}} d(z, y) \geq \min_{y \in \boldsymbol{z}} \text{dist}(z^{(k)}, y) \geq 0. \tag{3}$$

As the sequence $\boldsymbol{z}^{(k)}$ converges to $\boldsymbol{z}$, we also have $\lim_{k \to \infty} \text{dist}(\boldsymbol{z}^{(k)}, \boldsymbol{z}) = 0$. Therefore, using the inequalities (3), we obtain

$$\lim_{k \to \infty} \min_{y \in \boldsymbol{z}} \text{dist}(z^{(k)}, y) = 0.$$

As a direct consequence of the definition of an accumulation point z^* of the sequence $z^{(k)}$, we have $\min_{y \in \boldsymbol{z}} \text{dist}(z^*, y) = 0$. Finally, as $\boldsymbol{z}$ is closed inside $\mathbb{R}^n$, we have $z^* \in \boldsymbol{z}$.

  (ii) Whatever is $k \in \mathbb{N}$, the interval $\boldsymbol{z}^{(k)}$ being closed inside $\mathbb{R}^n$, there exists $z^{(k)} \in \boldsymbol{z}^{(k)}$ which satisfies

$$\text{dist}(z^{(k)}, z) = \min_{y \in \boldsymbol{z}^{(k)}} \text{dist}(y, z).$$

Similarly to the first case, we can prove that

$$\lim_{k \to \infty} \min_{y \in \boldsymbol{z}^{(k)}} \text{dist}(y, z) = 0.$$

Finally, $\lim_{k \to \infty} \text{dist}(z^{(k)}, z) = 0$ so we have constructed a sequence $z^{(k)} \in \boldsymbol{z}^{(k)}$ which converges to z. $\qquad \square$

Recall that by definition, given a set E and a subset F of E, F is closed inside E if and only if any point of E which is the limit of a converging sequence of points of F is also in F.

Proposition 4.3. *Let ϕ be a relation of $\mathbb{R}^n$. If the graph of ϕ is closed inside $\mathbb{R}^n$ then the graph of the AE-extension of ϕ is closed inside $\mathbb{KR}^n$.*

Proof. Consider any convergent sequence of intervals $(\boldsymbol{x}^{(k)})_{k \in \mathbb{N}}$ such that $\phi(\boldsymbol{x}^{(k)})$ holds for all $k \in \mathbb{N}$. We just have to prove that $\phi(\boldsymbol{x}^{(\infty)})$ holds, where $\boldsymbol{x}^{(\infty)}$ stands for the limit of the sequence $(\boldsymbol{x}^{(k)})_{k \in \mathbb{N}}$. We first pick up a subsequence $(\boldsymbol{y}^{(k)})_{k \in \mathbb{N}}$ of $(\boldsymbol{x}^{(k)})_{k \in \mathbb{N}}$ whose elements have constant componentwise proper/improper qualities. This is indeed possible, otherwise there would exists only a finite number of intervals for each possible 2^n proper/improper qualities of the components, that is a finite number of $\boldsymbol{x}^{(k)}$, which is absurd. As the subsequence converges to the same limit that the original sequence we just have to prove that $\phi(\boldsymbol{y}^{\infty})$ is true. Now, $\boldsymbol{y}^{(\infty)}$ has the same componentwise proper/improper qualities that the elements $\boldsymbol{y}^{(k)}$ because both $\mathbb{IR}$ and $\overline{\mathbb{IR}}$ are closed inside $\mathbb{KR}$. Define $P = P(\boldsymbol{y}^{(k)}) = P(\boldsymbol{y}^{(\infty)})$ and $I = I(\boldsymbol{y}^{(k)}) = I(\boldsymbol{y}^{(\infty)})$. We suppose that both P and I are not empty, the other cases being similar and simpler. Then, for all $k \in \mathbb{N}$, the following proposition is true:

$$\big(\forall x_{\mathcal{I}} \in \text{pro } \boldsymbol{y}_{\mathcal{I}}^{(k)}\big)\big(\exists x_{\mathcal{P}} \in \boldsymbol{y}_{\mathcal{P}}^{(k)}\big)\big(\phi(x)\big). \tag{4}$$

Now, by Lemma 4.1, for any $x_{\mathcal{I}} \in \text{pro } \boldsymbol{y}_{\mathcal{I}}^{(\infty)}$, there exists a sequence $x_{\mathcal{I}}^{(k)} \in \text{pro } \boldsymbol{y}_{\mathcal{I}}^{(k)}$ which converges to $x_{\mathcal{I}}$. Then using the quantified proposition (4), for all $k \in \mathbb{N}$ there exists $x_{\mathcal{P}}^{(k)} \in \boldsymbol{y}_{\mathcal{P}}^{(k)}$ such that $\phi(x^{(k)})$. The sequence $(x^{(k)})_{k \in \mathbb{N}}$ has at least one accumulation point $x^* \in \text{pro } \boldsymbol{y}^{(\infty)}$ by Lemma 4.1. Obviously, we have $x_{\mathcal{I}}^* = x_{\mathcal{I}}$. Furthermore, $\phi(x^*)$ holds because the graph of ϕ is closed inside $\mathbb{R}^n$. Therefore, we have eventually proved the following quantified proposition:

$$\big(\forall x_{\mathcal{I}} \in \text{pro } \boldsymbol{y}_{\mathcal{I}}^{(\infty)}\big)\big(\exists x_{\mathcal{P}} \in \boldsymbol{y}_{\mathcal{P}}^{(\infty)}\big)\big(\phi(x)\big).$$

Therefore that $\phi(\boldsymbol{y}^{(\infty)})$ is true. $\square$

5 AE-Extensions of Continuous Real Functions

The AE-extensions of continuous real functions are now defined. Their richer interpretations with respect to the classical interval extensions are obtained taking advantage of the additional degree of freedom offered by the proper/improper quality of generalized intervals.

5.1 Definition of AE-Extensions of Real Functions

The definition of extensions to classical intervals is first reformulated using AE-extensions of real relations. This new formulation of the definition of classical interval extensions will be applicable to generalized intervals, leading to the definition of AE-extensions. First recall the definition of extensions to classical intervals.

Definition 5.1 (Neumaier [25])**.** Consider a continuous real function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. An interval function $\boldsymbol{g} : \mathbb{IR}^n \longrightarrow \mathbb{IR}^m$ is an interval extension of f if and only if both following conditions are satisfied:

1. $\big(\forall x \in \mathbb{R}^n\big)\ \big(\boldsymbol{g}(x) = f(x)\big)$
2. $\big(\forall \boldsymbol{x} \in \mathbb{IR}^n\big)\ \big(\text{range}\,(f, \boldsymbol{x}) \subseteq \boldsymbol{g}(\boldsymbol{x})\big)$

Also, an interval function that only satisfies 2. is called a weak interval extension of f.

Remark 8. The following simplification is used: all the functions met in the sequel will be defined in $\mathbb{R}^n$. When other functions have to be considered (for example $1/x$ or $\ln(x)$) some attention should be given to the assocated domains of definition.

As illustrated by Example 4.3, the condition range $(f, \boldsymbol{x}) \subseteq \boldsymbol{g}(\boldsymbol{x})$ is equivalent to the quantified proposition

$$\big(\forall x \in \boldsymbol{x}\big)\big(\exists z \in \boldsymbol{g}(\boldsymbol{x})\big)\big(f(x) = z\big).$$

This quantified proposition is also equivalent to $\phi(\text{dual } \boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x}))$ where ϕ is the real relation defined in $\mathbb{R}^{n+m}$ by $\phi(x, z) \iff f(x) = z$. Therefore, the definition of the extensions to classical intervals can be reformulated in the following way:

Definition 5.2 (Reformulation of Definition 5.1)**.** Consider a continuous real function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. An interval function $\boldsymbol{g} : \mathbb{IR}^n \longrightarrow \mathbb{IR}^m$ is an interval extension of f if and only if both following conditions are satisfied:

1. $\big(\forall x \in \mathbb{R}^n\big)\,\big(\boldsymbol{g}(x) = f(x)\big)$
2. $\big(\forall \boldsymbol{x} \in \mathbb{IR}^n\big)\,\big(\phi(\text{dual } \boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x}))\big)$

where the real relation $\phi(x, z)$ is defined in $\mathbb{R}^{n+m}$ by $\phi(x, z) \iff f(x) = z$. Also, an interval function that only satisfies 2. is called a weak interval extension of f.

The definition of AE-extensions of real functions is eventually obtained extending the previous definition to generalized intervals.

Definition 5.3. Consider a continuous real function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. A generalized interval function $\boldsymbol{g} : \mathbb{KR}^n \longrightarrow \mathbb{KR}^m$ is an AE-extension of f if and only if both following conditions are satisfied:

1. $\big(\forall x \in \mathbb{R}^n\big)\,\big(\boldsymbol{g}(x) = f(x)\big)$
2. $\big(\forall \boldsymbol{x} \in \mathbb{KR}^n\big)\,\big(\phi(\text{dual } \boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x}))\big)$

where the real relation $\phi(x, z)$ is defined in $\mathbb{R}^{n+m}$ by $\phi(x, z) \iff f(x) = z$. Also, an interval function that only satisfies 2 is called a weak AE-extension of f.

Informally, when all involved intervals are proper, the definition of AE-extensions coincides with the one of interval extensions. When an improper interval is involved in place of a proper one, the related quantifier is changed, taking attention to keep the order AE inside the quantified proposition so obtained.

Formally, defining $\boldsymbol{z} = \boldsymbol{g}(\boldsymbol{x})$, $\boldsymbol{g}$ is an AE-extension of f if and only if the following quantified proposition is true:

$$\big(\forall x_{\mathcal{P}} \in \boldsymbol{x}_{\mathcal{P}}\big)\big(\forall z_{\mathcal{I}'} \in \text{pro } \boldsymbol{z}_{\mathcal{I}'}\big)\big(\exists z_{\mathcal{P}'} \in \boldsymbol{z}_{\mathcal{P}'}\big)\big(\exists x_{\mathcal{I}} \in \text{pro } \boldsymbol{x}_{\mathcal{I}}\big)\big(f(x) = z\big)$$

where $\mathcal{P} = \mathcal{P}(\boldsymbol{x})$, $\mathcal{I} = \mathcal{I}(\boldsymbol{x})$, $\mathcal{P}' = \mathcal{P}(\boldsymbol{z})$ and $\mathcal{I}' = \mathcal{I}(\boldsymbol{z})$. In the special case of real-valued function, i.e. $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, the previous quantified proposition can be written using the dependent quantifier introduced in Corollary 3.1 (page 141):

$$\big(\forall x_{\mathcal{P}} \in \boldsymbol{x}_{\mathcal{P}}\big)\big(\boldsymbol{q}^{(\boldsymbol{z})} z \in \text{pro } \boldsymbol{z}\big)\big(\exists x_{\mathcal{I}} \in \text{pro } \boldsymbol{x}_{\mathcal{I}}\big)\big(f(x) = z\big)$$

where $\boldsymbol{q}^{(\boldsymbol{z})}$ is defined by $\boldsymbol{q}^{(\boldsymbol{z})} = \exists$ if $\boldsymbol{z} \in \mathbb{IR}$ and $\boldsymbol{q}^{(\boldsymbol{z})} = \forall$ otherwise. The block $\big(\boldsymbol{q}^{(\boldsymbol{z})} z \in \text{pro } \boldsymbol{z}\big)$ is written at the center of the proposition in order to keep the succession AE in the proposition whatever is the quantifier $\boldsymbol{q}^{(\boldsymbol{z})}$, i.e. whatever is the proper/improper quality of $\boldsymbol{z}$.

The aim of the developments proposed hereafter is to construct generalized intervals $\boldsymbol{g}(\boldsymbol{x})$ that satisfy these latter propositions and to study their properties. Such generalized intervals $\boldsymbol{g}(\boldsymbol{x})$ are displayed by the following definition:

Definition 5.4. Consider a continuous real function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $\boldsymbol{x} \in \mathbb{KR}^n$ and $\boldsymbol{z} \in \mathbb{KR}^m$. The interval $\boldsymbol{z}$ is said to be *interpretable with respect to f and $\boldsymbol{x}$* (or shortly $(f, \boldsymbol{x})$-interpretable) if and only if $\phi(\text{dual } \boldsymbol{x}, \boldsymbol{z})$ is true, where the real relation $\phi(x, z)$ is defined in $\mathbb{R}^{n+m}$ by $\phi(x, z) \iff f(x) = z$.

As a consequence of this definition, AE-extensions of f are defined so as to construct $(f, \boldsymbol{x})$-interpretable intervals. The next subsection provides some examples of interpretations of $(f, \boldsymbol{x})$-interpretable intervals.

5.2   Some Interpretations of AE-Extensions

The AE-extensions can be used for many purposes: they can either be used to compute inner or outer approximations of functions ranges over boxes, or prove that a box is inside the projection of a relation $f(x) = z_0$. Some examples of these interpretations are now provided. A continuous function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and two intervals $\boldsymbol{x} \in \mathbb{KR}$ and $\boldsymbol{y} \in \mathbb{KR}$ are considered in this section. The interpretations of some $(f, \boldsymbol{x}, \boldsymbol{y})$-interpretable intervals $\boldsymbol{z} \in \mathbb{KR}$ are investigated. Also, these examples will show that the generalized interval inclusion allows comparing the accuracy of $(f, \boldsymbol{x}, \boldsymbol{y})$-interpretable intervals. This use of the generalized interval inclusion will be formalized in Section 6.

Suppose that $\boldsymbol{x} \in \mathbb{IR}$ and $\boldsymbol{y} \in \mathbb{IR}$

In this case, $\boldsymbol{z}$ has to be proper. If $\boldsymbol{z}$ was not proper, i.e. was improper and non-degenerate, a quantified proposition

$$\big(\forall x \in \boldsymbol{x}\big)\big(\forall y \in \boldsymbol{y}\big)\big(\forall z \in \text{pro } \boldsymbol{z}\big)\big(f(x, y) = z\big)$$

would have to be true, with pro $\boldsymbol{z}$ non-degenerate, which is absurd (see Proposition 5.1 for a formal argument). Therefore $\boldsymbol{z}$ is $(f, \boldsymbol{x}, \boldsymbol{y})$-interpretable if and only if

$$\big(\forall x \in \boldsymbol{x}\big)\big(\forall y \in \boldsymbol{y}\big)\big(\exists z \in \boldsymbol{z}\big)\big(f(x) = z\big),$$

that is $\boldsymbol{z} \supseteq \text{range}\,(f, \boldsymbol{x}, \boldsymbol{y})$. Consider two $(f, \boldsymbol{x})$-interpretable intervals $\boldsymbol{z}$ and $\boldsymbol{z}'$ related by $\boldsymbol{z}' \subseteq \boldsymbol{z}$. So, both following quantified propositions are true:

$$\big(\forall x \in \boldsymbol{x}\big)\big(\forall y \in \boldsymbol{y}\big)\big(\exists z \in \boldsymbol{z}\big)\big(f(x, y) = z\big)$$

$$\big(\forall x \in \boldsymbol{x}\big)\big(\forall y \in \boldsymbol{y}\big)\big(\exists z \in \boldsymbol{z}'\big)\big(f(x, y) = z\big).$$

It can be noticed that, because $\boldsymbol{z}' \subseteq \boldsymbol{z}$, the first quantified proposition provides less information than the second one (because the second implies the first). So $\boldsymbol{z}'$ can be considered as more accurate than $\boldsymbol{z}$. In this case, the comparison between $(f, \boldsymbol{x}, \boldsymbol{y})$-interpretable intervals is similar to the classical interval extension context.

Suppose that $\boldsymbol{x} \in \overline{\mathbb{IR}}$ and $\boldsymbol{y} \in \overline{\mathbb{IR}}$

In this case, a $(f, \boldsymbol{x}, \boldsymbol{y})$-interpretable interval can either be proper or improper. Consider two $(f, \boldsymbol{x}, \boldsymbol{y})$-interpretable intervals $\boldsymbol{z} \in \mathbb{IR}$ and $\boldsymbol{z}' \in \overline{\mathbb{IR}}$. So, both following quantified propositions are true:

$$\big(\exists z \in \boldsymbol{z}\big)\big(\exists x \in \text{pro } \boldsymbol{x}\big)\big(\exists y \in \text{pro } \boldsymbol{y}\big)\big(f(x, y) = z\big)$$

$$\big(\forall z \in \text{pro } \boldsymbol{z}'\big)\big(\exists x \in \text{pro } \boldsymbol{x}\big)\big(\exists y \in \text{pro } \boldsymbol{y}\big)\big(f(x, y) = z\big).$$

So, $\boldsymbol{z} \bigcap \mathrm{range}\,(f, \mathrm{pro}\ \boldsymbol{x}, \mathrm{pro}\ \boldsymbol{y}) \neq \emptyset$ and pro $\boldsymbol{z}' \subseteq \mathrm{range}\,(f, \mathrm{pro}\ \boldsymbol{x}, \mathrm{pro}\ \boldsymbol{y})$ are the two possible interpretations of a $(f, \boldsymbol{x}, \boldsymbol{y})$-interpretable interval in this case. Now, consider a third $(f, \boldsymbol{x}, \boldsymbol{y})$ interpretable interval $\boldsymbol{z}'' \in \overline{\mathbb{IR}}$ and suppose that $\boldsymbol{z}'' \subseteq \boldsymbol{z}' \subseteq \boldsymbol{z}$. So, the next quantified proposition is also true.

$$\big(\forall z \in \mathrm{pro}\ \boldsymbol{z}''\big)\big(\exists x \in \mathrm{pro}\ \boldsymbol{x}\big)\big(\exists y \in \mathrm{pro}\ \boldsymbol{y}\big)\big(f(x,y) = z\big).$$

On one hand, $\boldsymbol{z}'' \subseteq \boldsymbol{z}'$ implies pro $\boldsymbol{z}' \subseteq$ pro $\boldsymbol{z}''$ and therefore the second quantified proposition provides less information than the third one (because the third implies the second). On the other hand, $\boldsymbol{z}' \subseteq \boldsymbol{z}$ implies pro $\boldsymbol{z}' \bigcap \boldsymbol{z} \neq \emptyset$ and therefore the first quantified proposition provides less information than the second one (because the second implies the first). So, the interval $\boldsymbol{z}''$ can be considered as more accurate than $\boldsymbol{z}'$, and $\boldsymbol{z}'$ as more accurate than $\boldsymbol{z}$. The inclusion between generalized intervals can be used to model the accuracy of $(f, \boldsymbol{x}, \boldsymbol{y})$-interpretable intervals.

Suppose that $\boldsymbol{x} \in \mathbb{IR}$ and $\boldsymbol{y} \in \overline{\mathbb{IR}}$

In this case, a $(f, \boldsymbol{x})$-interpretable interval can be either proper or improper. As before, consider two $(f, \boldsymbol{x})$-interpretable intervals $\boldsymbol{z} \in \mathbb{IR}$ and $\boldsymbol{z}' \in \overline{\mathbb{IR}}$. So, both following quantified propositions are true:

$$\big(\forall x \in \boldsymbol{x}\big)\big(\exists z \in \boldsymbol{z}\big)\big(\exists y \in \mathrm{pro}\ \boldsymbol{y}\big)\big(f(x,y) = z\big)$$

$$\big(\forall x \in \boldsymbol{x}\big)\big(\forall z \in \mathrm{pro}\ \boldsymbol{z}'\big)\big(\exists y \in \mathrm{pro}\ \boldsymbol{y}\big)\big(f(x,y) = z\big).$$

If the first one does not offer any interesting interpretation, the second means that the interval $\boldsymbol{x}$ is a subset of the projection of the relation $f(x,y) = z_0$ on the x-axis whatever is $z_0 \in \mathrm{pro}\ \boldsymbol{z}'$. As in the previous case, if $\boldsymbol{z}' \subseteq \boldsymbol{z}$ then pro $\boldsymbol{z}' \bigcap \boldsymbol{z} \neq \emptyset$ and therefore the first quantified proposition provides less information than the second. Once more, the inclusion between generalized intervals can be used to model the accuracy of $(f, \boldsymbol{x}, \boldsymbol{y})$-interpretable intervals.

5.3 Some properties of AE-extensions

Some general properties of AE-extensions are now investigated. First of all, the next proposition states formally that when restricted to proper interval arguments AE-extensions coincide with the extensions to classical intervals, i.e. the image of a proper interval is a proper interval that contains the range of the function.

Proposition 5.1. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous function, $\boldsymbol{x} \in \mathbb{IR}^n$ a proper interval and $\boldsymbol{z} \in \mathbb{KR}^m$. The interval $\boldsymbol{z}$ is $(f, \boldsymbol{x})$-interpretable if and only if*

$$\boldsymbol{z} \supseteq \square\,range\,(f, \boldsymbol{x})\,,$$

which implies in particular that $\boldsymbol{z}$ is proper.

Proof. $\boldsymbol{z} \supseteq \square\,\mathrm{range}\,(f, \boldsymbol{x})$ obviously implies that $\boldsymbol{z}$ is $(f, \boldsymbol{x})$-interpretable. It remains to prove that if $\boldsymbol{z}$ is $(f, \boldsymbol{x})$-interpretable then $\boldsymbol{z} \supseteq \square\,\mathrm{range}\,(f, \boldsymbol{x})$. By definition, $\boldsymbol{z}$ satisfies

$$\big(\forall x \in \boldsymbol{x}\big)\big(\forall z_{\mathcal{I}} \in \mathrm{pro}\ \boldsymbol{z}_{\mathcal{I}}\big)\big(\exists z_{\mathcal{P}} \in \boldsymbol{z}_{\mathcal{P}}\big)\big(f(x) = z\big).$$

The latter quantified proposition obviously implies the following one:

$$\big(\forall x \in \boldsymbol{x}\big)\big(\forall z_{\mathcal{I}} \in \mathrm{pro}\ \boldsymbol{z}_{\mathcal{I}}\big)\big(f_{\mathcal{I}}(x) = z_{\mathcal{I}}\big).$$

Now, suppose that $\boldsymbol{z}_{\mathcal{I}}$ is not degenerate. So there exists $z_{\mathcal{I}} \in \text{pro } \boldsymbol{z}_{\mathcal{I}}$ and $z'_{\mathcal{I}} \in \text{pro } \boldsymbol{z}_{\mathcal{I}}$ which satisfies $z_{\mathcal{I}} \neq z'_{\mathcal{I}}$. So, for any $x \in \boldsymbol{x}$ we have $f_{\mathcal{I}}(x) = z_{\mathcal{I}}$ and $f_{\mathcal{I}}(x) = z'_{\mathcal{I}}$, which is absurd. So $\boldsymbol{z}_{\mathcal{I}}$ is degenerate and thus also proper. Therefore $\boldsymbol{z}$ has to be proper. As it is also $(f, \boldsymbol{x})$-interpretable, it satisfies $(\forall x \in \boldsymbol{x})(\exists z \in \boldsymbol{z})(f(x) = z)$, that is $\boldsymbol{z} \supseteq \square \text{range}\,(f, \boldsymbol{x})$. $\hfill\square$

Then, a question arises: does any continuous real function have at least one AE-extension? The next proposition provides a positive answer to this question.

Proposition 5.2. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous function and $\boldsymbol{r} : \mathbb{KR}^n \longrightarrow \mathbb{IR}^m$ defined by $\boldsymbol{r}(\boldsymbol{x}) = \square \text{range}\,(f, \text{pro } \boldsymbol{x})$. Then, $\boldsymbol{r}$ is an AE-extension of f.*

Proof. First notice that $\boldsymbol{r}(x) = f(x)$ for all $x \in \mathbb{R}^n$. Denote the relation $f(x) = z$ by $\phi(x, z)$ and consider any $\boldsymbol{x} \in \mathbb{KR}^n$. As $\boldsymbol{r}(\boldsymbol{x}) = \square \text{range}\,(f, \text{pro } \boldsymbol{x})$, the following quantified proposition is true:

$$\big(\forall x \in \text{pro } \boldsymbol{x}\big)\big(\exists z \in \boldsymbol{r}(\boldsymbol{x})\big)\big(f(x) = z\big).$$

That is, $\phi(\text{imp } \boldsymbol{x}, \boldsymbol{r}(\boldsymbol{x}))$ holds (as $\text{dual}\,(\text{pro } \boldsymbol{x}) = \text{imp } \boldsymbol{x}$). Now, as the inclusion $(\text{imp } \boldsymbol{x}) \subseteq (\text{dual } \boldsymbol{x})$ holds for any $\boldsymbol{x} \in \mathbb{KR}^n$, the Proposition 4.1 can be applied so as to prove that $\phi(\text{dual } \boldsymbol{x}, \boldsymbol{r}(\boldsymbol{x}))$ holds. Therefore, the generalized interval $\boldsymbol{r}(\boldsymbol{x})$ is $(f, \boldsymbol{x})$-interpretable. $\hfill\square$

Example 5.1. Consider a two variables function f and two generalized intervals $\boldsymbol{x}_1 \in \mathbb{IR}$ and $\boldsymbol{x}_2 \in \overline{\mathbb{IR}}$. Define $\boldsymbol{z} = \square \text{range}\,(f, \boldsymbol{x}_1, \text{pro } \boldsymbol{x}_2)$. Then, by construction of $\boldsymbol{z}$ the quantified proposition

$$\big(\forall x_1 \in \boldsymbol{x}_1\big)\big(\forall x_2 \in \text{pro } \boldsymbol{x}_2\big)\big(\exists z \in \boldsymbol{z}\big)\big(f(x) = z\big)$$

is true. It obviously implies

$$\big(\forall x_1 \in \boldsymbol{x}_1\big)\big(\exists x_2 \in \text{pro } \boldsymbol{x}_2\big)\big(\exists z \in \boldsymbol{z}\big)\big(f(x) = z\big).$$

Therefore, the interval $\boldsymbol{z}$ is $(f, \boldsymbol{x}_1, \boldsymbol{x}_2)$-interpretable.

Finally, the next proposition gives a lower bound (in the sense of the inclusion) for any AE-extension. It will be useful for the coming developments.

Proposition 5.3. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous function, $\boldsymbol{x} \in \mathbb{KR}^n$ and $\boldsymbol{z} \in \mathbb{KR}^m$ be a $(f, \boldsymbol{x})$-interpretable. Then $\text{imp } \boldsymbol{r}(\boldsymbol{x}) \subseteq \boldsymbol{z}$ holds, where $\boldsymbol{r}(\boldsymbol{x}) = \square \text{range}\,(f, \text{pro } \boldsymbol{x})$.*

Proof. Denote the relation $f(x) = z$ by $\phi(x, z)$. By definition, the interval $\boldsymbol{z}$ is $(f, \boldsymbol{x})$-interpretable implies that $\phi(\text{dual } \boldsymbol{x}, \boldsymbol{z})$ is true. Then, define $\boldsymbol{y}$ by $\boldsymbol{y} = \square\{(x, z) \in (\text{pro } \boldsymbol{x}, \text{pro } \boldsymbol{z}) | \phi(x, z)\}$, which is well defined by the first assertion of Proposition 4.2. We obviously have $\boldsymbol{y} \subseteq (\text{pro } \boldsymbol{x}, \boldsymbol{r}(\boldsymbol{x}))$. As $\boldsymbol{r}(\boldsymbol{x})$ and $\boldsymbol{y}$ are proper, dualizing both sides of this inclusion raises $(\text{imp } \boldsymbol{x}, \text{imp } \boldsymbol{r}(\boldsymbol{x})) \subseteq \text{imp } \boldsymbol{y}$. Furthermore, applying the second assertion of the Proposition 4.2, we prove the $\phi(\text{dual } \boldsymbol{x}, \boldsymbol{z}) \Longrightarrow (\text{imp } \boldsymbol{y}) \subseteq (\text{dual } \boldsymbol{x}, \boldsymbol{z})$. Combining both previous inclusions raises $(\text{imp } \boldsymbol{x}, \text{imp } \boldsymbol{r}(\boldsymbol{x})) \subseteq (\text{dual } \boldsymbol{x}, \boldsymbol{z})$, which eventually implies $(\text{imp } \boldsymbol{r}(\boldsymbol{x})) \subseteq \boldsymbol{z}$ (notice that $\text{imp } \boldsymbol{x} \subseteq \text{dual } \boldsymbol{x}$ is true whatever is $\boldsymbol{x}$). $\hfill\square$

6   Minimal AE-Extensions

In the context of classical interval extensions, if two extensions $\boldsymbol{g}$ and $\boldsymbol{h}$ of a real function f are related by the inclusion $\boldsymbol{g}(\boldsymbol{x}) \subseteq \boldsymbol{h}(\boldsymbol{x})$, then $\boldsymbol{g}$ is more accurate than $\boldsymbol{h}$. This is justified because once $\boldsymbol{g}(\boldsymbol{x})$ is evaluated, $\boldsymbol{h}(\boldsymbol{x})$ does not give any additional information. Or equivalently, because if $\boldsymbol{g}$ is an interval extension of f then any interval function $\boldsymbol{h}$ which satisfies $\boldsymbol{g}(\boldsymbol{x}) \subseteq \boldsymbol{h}(\boldsymbol{x})$ is also an extension of f. Subsection 5.2 has illustrated that this comparison between the accuracy of two classical interval extensions can be carried to AE-extensions, leading to the following definition:

Definition 6.1. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous real function and $\boldsymbol{x} \in \mathbb{KR}^n$. Consider two $(f, \boldsymbol{x})$-interpretable intervals $\boldsymbol{z} \in \mathbb{KR}^m$ and $\boldsymbol{z}' \in \mathbb{KR}^m$. Then $\boldsymbol{z}$ is more accurate than $\boldsymbol{z}'$ if and only if $\boldsymbol{z} \subseteq \boldsymbol{z}'$. Also, $\boldsymbol{z}$ is strictly more accurate than $\boldsymbol{z}'$ if furthermore $\boldsymbol{z} \neq \boldsymbol{z}'$.

This definition is justified by the following proposition, which generalizes the informal ideas presented in Subsection 5.2:

Proposition 6.1. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous real function and $\boldsymbol{x} \in \mathbb{KR}^n$. Consider a $(f, \boldsymbol{x})$-interpretable intervals $\boldsymbol{z} \in \mathbb{KR}^m$ and an interval $\boldsymbol{z}' \in \mathbb{KR}^m$. If $\boldsymbol{z} \subseteq \boldsymbol{z}'$ then $\boldsymbol{z}'$ is also $(f, \boldsymbol{x})$-interpretable.*

Proof. Define the relation ϕ by $\phi(x, z) \iff f(x) = z$. So, the interval $\boldsymbol{z}$ is $(f, \boldsymbol{x})$-interpretable if and only if $\phi(\text{dual } \boldsymbol{x}, \boldsymbol{z})$ is true and the interval $\boldsymbol{z}'$ is $(f, \boldsymbol{x})$-interpretable if and only if $\phi(\text{dual } \boldsymbol{x}, \boldsymbol{z}')$ is true. Now, as $\boldsymbol{z} \subseteq \boldsymbol{z}'$ and because dual $\boldsymbol{x} \subseteq$ dual $\boldsymbol{x}$, we have $(\text{dual } \boldsymbol{x}, \boldsymbol{z})^T \subseteq (\text{dual } \boldsymbol{x}, \boldsymbol{z}')^T$. Therefore, the Proposition 4.1 proves that $\phi(\text{dual } \boldsymbol{x}, \boldsymbol{z}) \implies \phi(\text{dual } \boldsymbol{x}, \boldsymbol{z}')$. Finally, if $\boldsymbol{z}$ is $(f, \boldsymbol{x})$-interpretable then $\boldsymbol{z}'$ is also $(f, \boldsymbol{x})$-interpretable. $\qquad\square$

Example 6.1. Consider $f(x, y) = x + y$ and $\boldsymbol{x} = [-1, 1]$ and $\boldsymbol{y} = [5, 2]$. The interval $\boldsymbol{z} = [4, 3]$ can be proved to be $(f, \boldsymbol{x}, \boldsymbol{y})$-interpretable[2]. Therefore, the following quantified proposition is true:

$$\big(\forall x \in \boldsymbol{x}\big)\big(\forall z \in [3, 4]\big)\big(\exists y \in \boldsymbol{y}\big)\big(x + y = z\big)$$

Using the generalized interval inclusion, one can raise less accurate $(f, \boldsymbol{x}, \boldsymbol{y})$-interpretable intervals. E.g. $\boldsymbol{z} \subseteq [3.9, 3.1]$ so the following quantified proposition is true:

$$\big(\forall x \in \boldsymbol{x}\big)\big(\forall z \in [3.1, 3.9]\big)\big(\exists y \in \boldsymbol{y}\big)\big(x + y = z\big)$$

Clearly the latter quantified proposition provides less information than the former. Also $\boldsymbol{z} \subseteq [3.9, 3.1] \subseteq [3.5, 5]$ so the following quantified proposition is true:

$$\big(\forall x \in \boldsymbol{x}\big)\big(\exists z \in [3.5, 5]\big)\big(\exists y \in \boldsymbol{y}\big)\big(x + y = z\big)$$

Once more, the latter quantifier proposition provides less information than the firsts two.

Once Definition 6.1 is stated, a definition of minimal AE-extensions can be proposed in the following natural way: an AE-extension is minimal if and only if there does not exist any AE-extension which would be strictly more accurate.

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[2]Actually, $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$ using Kaucher arithmetic. See Section 10 where Kaucher arithmetic is proved to raise $(f, \boldsymbol{x}, \boldsymbol{y})$-interpretable intervals.

Definition 6.2. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous function and $\boldsymbol{x} \in \mathbb{KR}^n$. A $(f, \boldsymbol{x})$-interpretable interval $\boldsymbol{z} \in \mathbb{KR}^m$ is minimal if and only if for any $(f, \boldsymbol{x})$-interpretable interval $\boldsymbol{z}' \in \mathbb{KR}^m$,

$$\boldsymbol{z}' \subseteq \boldsymbol{z} \implies \boldsymbol{z}' = \boldsymbol{z}.$$

An AE-extension $\boldsymbol{g} : \mathbb{KR}^n \longrightarrow \mathbb{KR}^m$ of f is minimal if and only if for all $\boldsymbol{x} \in \mathbb{KR}^n$ the $(f, \boldsymbol{x})$-interpretable interval $\boldsymbol{g}(\boldsymbol{x})$ is minimal.

In the special cases where $\boldsymbol{x}$ is either proper or improper, the minimality of the AE-extensions is related to the minimality (respectively maximality) of the outer (respectively inner) approximations of the range of the extended function:

Proposition 6.2. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous function and $\boldsymbol{x} \in \mathbb{KR}^n$ and $\boldsymbol{z} \in \mathbb{KR}^m$.*

(i) *Suppose $\boldsymbol{x} \in \mathbb{IR}^n$. In this case, the interval $\boldsymbol{z}$ is a minimal $(f, \boldsymbol{x})$-interpretable interval if and only if $\boldsymbol{z} = \square\, \mathrm{range}\,(f, \boldsymbol{x})$, i.e. the unique minimal outer approximation of $\mathrm{range}\,(f, \boldsymbol{x})$.*

(ii) *Suppose $\boldsymbol{x} \in \overline{\mathbb{IR}}^n$. In this case, if the interval $\boldsymbol{z}$ is improper and $\mathrm{pro}\,\boldsymbol{z}$ is an maximal inner approximation of the range of f over $\mathrm{pro}\,\boldsymbol{x}$ then $\boldsymbol{z}$ is an minimal $(f, \boldsymbol{x})$-interpretable interval.*

Proof. (i) As proved in Subsection 5.2, if $\boldsymbol{x}$ is proper then $\boldsymbol{z}$ is $(f, \boldsymbol{x})$-interpretable if and only if it is both proper and an outer approximation of $\mathrm{range}\,(f, \boldsymbol{x})$. Therefore, $\boldsymbol{z}$ is a minimal $(f, \boldsymbol{x})$-interpretable interval if and only if $\boldsymbol{z} = \square\, \mathrm{range}\,(f, \boldsymbol{x})$.

(ii) Consider any $(f, \boldsymbol{x})$-interpretable interval $\boldsymbol{z}' \in \mathbb{KR}^m$ which satisfies $\boldsymbol{z}' \subseteq \boldsymbol{z}$. We just have to prove that $\boldsymbol{z}' = \boldsymbol{z}$. Due to $\boldsymbol{z}' \subseteq \boldsymbol{z} \in \overline{\mathbb{IR}}$, $\boldsymbol{z}'$ is also improper. Therefore, $\mathrm{pro}\,\boldsymbol{z}'$ is an inner approximation of $\mathrm{range}\,(f, \mathrm{pro}\,\boldsymbol{x})$ and $\mathrm{pro}\,\boldsymbol{z} \subseteq \mathrm{pro}\,\boldsymbol{z}'$. So by the maximality of the inner approximation $\mathrm{pro}\,\boldsymbol{z}$, we get $\mathrm{pro}\,\boldsymbol{z}' = \mathrm{pro}\,\boldsymbol{z}$, and eventually $\boldsymbol{z}' = \boldsymbol{z}$. $\qquad\square$

Remark 9. The counterpart of the case (ii) of Proposition 6.2 is false. I.e. there exists some quantified proposition

$$\bigl(\forall z_1 \in \boldsymbol{z}_1\bigr)\bigl(\exists z_2 \in \boldsymbol{z}_2\bigr)\bigl(\exists x_1 \in \boldsymbol{x}_1\bigr)\bigl(\exists x_2 \in \boldsymbol{x}_2\bigr)\bigl(z = f(x)\bigr)$$

where neither $\boldsymbol{z}_1$ can be enlarged nor $\boldsymbol{z}_2$ can be retracted keeping the truth of the proposition. Therefore minimal $(f, \boldsymbol{x})$-interpretable intervals with $\boldsymbol{x} \in \overline{\mathbb{IR}}^n$ can be non-improper.

Proposition 6.2 allows the construction of some minimal AE-extension of a one variable real-valued function $f : \mathbb{R} \longrightarrow \mathbb{R}$.

Example 6.2. Consider the function $\exp(x)$. As it is a one variable function, the interval $\boldsymbol{x} \in \mathbb{KR}$ is either proper or improper and the Proposition 6.2 can be applied for any interval argument $\boldsymbol{x}$. Define the interval function $\exp^* : \mathbb{KR} \longrightarrow \mathbb{KR}$ in the following way:

- if $\boldsymbol{x} \in \mathbb{IR}$, then

$$\exp^*(\boldsymbol{x}) = \mathrm{range}\,(\exp, \boldsymbol{x}) = [\exp(\underline{x}), \exp(\overline{x})].$$

- if $\boldsymbol{x} \in \overline{\mathbb{IR}}$, then

$$\exp^*(\boldsymbol{x}) = \mathrm{dual}\,\mathrm{range}\,(\exp, \mathrm{pro}\,\boldsymbol{x}) = [\exp(\underline{x}), \exp(\overline{x})].$$

The general expression of $\exp^*$ is therefore $\exp^*(\boldsymbol{x}) = [\exp(\underline{x}), \exp(\overline{x})]$. Then Proposition 6.2 proves that $\exp^*$ is a minimal AE-extension of exp. It will be proved in Section 9 that there does not exist any other minimal AE-extension of the function exp. In the same way, one can compute a minimal AE-extension of $\ln x$: for any $\boldsymbol{x} \in \mathbb{KR}$ such that $\boldsymbol{x} > 0$, we have $\ln^* \boldsymbol{x} = [\ln \underline{x}, \ln \overline{x}]$.

Remark 10. It can be noticed that the minimal AE-extensions of exp and ln coincide with their extensions defined by Kaucher (see Subsection 2 page 138). This coincidence will be proved to happen for any one variable function in Section 10. In this case, both extensions f^* and $f^{\mathbb{KR}}$ will be denoted by the symbol f.

A question then arises: does always exist a minimal AE-extension which is more accurate than a given AE-extension? The next proposition gives a positive answer to this question. First of all, the next technical result is needed.

Lemma 6.1. *Let $E \subseteq \mathbb{KR}^n$ be non-empty, closed and bounded. Then E contains at least one inclusion minimal element, i.e. an element which is not included in any other element of E.*

Proof. Consider the order preserving homeomorphism $\sigma : \mathbb{KR}^n \longrightarrow \mathbb{R}^{2n}$ introduced in [32], i.e.

$$\sigma(\boldsymbol{x}) = (-\inf \boldsymbol{x}_1, ..., -\inf \boldsymbol{x}_n, \sup \boldsymbol{x}_1, ..., \sup \boldsymbol{x}_n).$$

As σ is order preserving, if x^* is a minimal element of $\sigma(E)$ then $\sigma^{-1}(x^*)$ is a minimal element of E. E being closed in the complete space $\mathbb{KR}^n$ and bounded, it is compact. Finally as σ is continuous, $\sigma(E)$ is compact and nonempty. Therefore, $\sigma(E)$ have at least one minimal element . $\qquad\square$

Proposition 6.3. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous function and $\boldsymbol{g}$ be an AE-extension of f. Then there exists an minimal AE-extension of f which is more accurate than $\boldsymbol{g}$.*

Proof. We just have to prove that for any $\boldsymbol{x} \in \mathbb{KR}^n$, there exists a minimal $(f, \boldsymbol{x})$-interpretable interval which is more accurate than $\boldsymbol{g}(\boldsymbol{x})$. So, consider any $\boldsymbol{x} \in \mathbb{KR}^n$ and define $E = \{\boldsymbol{z} | \boldsymbol{z} \subseteq \boldsymbol{g}(\boldsymbol{x})\}$ and $F = \{\boldsymbol{z} \in \mathbb{KR}^m | \phi(\text{dual } \boldsymbol{x}, \boldsymbol{z})\}$ where $\phi(x, z) \iff f(x) = z$. Therefore, the intervals of $E \bigcap F$ are both $(f, \boldsymbol{x})$-interpretable and more accurate than $\boldsymbol{g}(\boldsymbol{x})$. On one hand, E is obviously closed. On the other hand, the graph of ϕ is closed inside $\mathbb{R}^n$ because f is continuous and therefore F is closed thanks Proposition 4.3 (page 145). Therefore, $E \bigcap F$ is closed. Furthermore, $E \bigcap F$ is bounded thanks to Proposition 5.3 (page 150) which proves that $\boldsymbol{z} \in E \bigcap F$ implies dual $\square \text{range}(f, \text{pro } \boldsymbol{x}) \subseteq \boldsymbol{z} \subseteq \boldsymbol{g}(\boldsymbol{x})$. Finally, $E \bigcap F$ is not empty because $\boldsymbol{g}(\boldsymbol{x}) \in E \bigcap F$. Therefore, we can apply Lemma 6.1 which proves that $E \bigcap F$ has at least one minimal element, which is an minimal $(f, \boldsymbol{x})$-interpretable interval by definition of E of F. $\qquad\square$

In particular, Proposition 6.3 proves that any box included inside the range of a continuous function $\text{range}(f, \boldsymbol{x})$ can be extended to a box which is a maximal inner approximation. It may happen that a function has several different minimal AE-extensions, in particular when the AE-extensions are used to compute inner approximations of the ranges of vector-valued functions. The next proposition gives a sufficient condition for an interval function to be the *unique* minimal AE-extension of a continuous real function.

Proposition 6.4. *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *be a continuous function and* $\boldsymbol{x} \in \mathbb{KR}^n$. *The interval* $\boldsymbol{z} \in \mathbb{KR}^m$ *is the unique minimal* $(f, \boldsymbol{x})$-*interpretable interval if and only if for any* $\boldsymbol{z}' \in \mathbb{KR}^m$,

$$\boldsymbol{z}' \text{ is } (f, \boldsymbol{x})\text{-interpretable} \iff \boldsymbol{z} \subseteq \boldsymbol{z}'.$$

Proof. ($\Longrightarrow$) Suppose that $\boldsymbol{z}$ is the unique minimal $(f, \boldsymbol{x})$-interpretable interval. On one hand, by the Proposition 6.1, we have $\boldsymbol{z} \subseteq \boldsymbol{z}'$ implies that $\boldsymbol{z}'$ is $(f, \boldsymbol{x})$-interpretable. On the other hand, consider any $\boldsymbol{z}'$ which is $(f, \boldsymbol{x})$-interpretable. Then, by the Proposition 6.3, there exists an minimal $(f, \boldsymbol{x})$-interpretable interval $\boldsymbol{z}'' \in \mathbb{KR}^m$ which satisfies $\boldsymbol{z}'' \subseteq \boldsymbol{z}'$. Finally, $\boldsymbol{z}'' = \boldsymbol{z}$ because $\boldsymbol{z}$ is the unique minimal $(f, \boldsymbol{x})$-interpretable interval and we have proved $\boldsymbol{z} \subseteq \boldsymbol{z}'$.

($\Longleftarrow$) Suppose that $\boldsymbol{z}'$ is $(f, \boldsymbol{x})$-interpretable if and only if $\boldsymbol{z} \subseteq \boldsymbol{z}'$. In particular, $\boldsymbol{z} \subseteq \boldsymbol{z}$ implies that $\boldsymbol{z}$ is $(f, \boldsymbol{x})$-interpretable. Consider any $(f, \boldsymbol{x})$-interpretable $\boldsymbol{z}' \in \mathbb{KR}^n$ which satisfies $\boldsymbol{z}' \subseteq \boldsymbol{z}$. By hypothesis $\boldsymbol{z} \subseteq \boldsymbol{z}'$ because $\boldsymbol{z}'$ is $(f, \boldsymbol{x})$-interpretable. So we have $\boldsymbol{z}' = \boldsymbol{z}$. Therefore, by the definition of $\boldsymbol{z}$ minimality, $\boldsymbol{z}$ is minimal. Finally consider any minimal $(f, \boldsymbol{x})$-interpretable $\boldsymbol{z}' \in \mathbb{KR}^n$. As $\boldsymbol{z}'$ is $(f, \boldsymbol{x})$-interpretable, we have by hypothesis $\boldsymbol{z} \subseteq \boldsymbol{z}'$ and, because $\boldsymbol{z}'$ is supposed to be minimal, the definition of minimality implies $\boldsymbol{z} = \boldsymbol{z}'$. $\square$

In particular, the previous proposition will be used in Section 9 in order to prove that any continuous real-valued function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ has an unique minimal AE-extension. The next proposition provides an interesting property which is available when the uniqueness of the minimal $(f, \boldsymbol{x})$-interpretable interval is proved: like in the case of classical interval extensions (where minimal extensions are always unique) one can construct a $(f, \boldsymbol{x})$-interpretable interval intersecting two $(f, \boldsymbol{x})$-interpretable intervals.

Proposition 6.5. *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *be a continuous function and* $\boldsymbol{x} \in \mathbb{KR}^n$. *Suppose that* $\boldsymbol{z}^* \in \mathbb{KR}^m$ *is the unique minimal* $(f, \boldsymbol{x})$-*interpretable interval and consider two* $(f, \boldsymbol{x})$-*interpretable intervals* $\boldsymbol{z}, \boldsymbol{z}' \in \mathbb{KR}^m$. *Then the interval* $\boldsymbol{z} \wedge \boldsymbol{z}'$ *is also* $(f, \boldsymbol{x})$-*interpretable (and is obviously more accurate than* $\boldsymbol{z}$ *and* $\boldsymbol{z}'$).

Proof. By Proposition 6.4, we have both $\boldsymbol{z}^* \subseteq \boldsymbol{z}$ and $\boldsymbol{z}^* \subseteq \boldsymbol{z}'$. Therefore, $\boldsymbol{z}^* \subseteq \boldsymbol{z} \wedge \boldsymbol{z}'$ which eventually implies that $\boldsymbol{z} \wedge \boldsymbol{z}'$ is $(f, \boldsymbol{x})$-interpretable. $\square$

Note that this property is generally not true when there is no unique minimal $(f, \boldsymbol{x})$-interpretable interval, e.g. when $\boldsymbol{x}$, $\boldsymbol{z}$ and $\boldsymbol{z}'$ are improper so pro $\boldsymbol{z}$ and pro $\boldsymbol{z}'$ are inner approximations of range $(f, \text{pro } \boldsymbol{x})$ while pro $(\boldsymbol{z} \wedge \boldsymbol{z}') = (\text{pro } \boldsymbol{z}) \vee (\text{pro } \boldsymbol{z}')$ is generally not an inner approximation of range $(f, \text{pro } \boldsymbol{x})$.

Finally, the next proposition is a technical result which will be used in Section 7. First, the following lemma has to be established:

Lemma 6.2. *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *be a continuous function,* $\boldsymbol{x} \in \mathbb{KR}^n$ *and* $\boldsymbol{z}$ *be an minimal* $(f, \boldsymbol{x})$-*interpretable interval. Then,* $\boldsymbol{z} \subseteq \boldsymbol{r}(\boldsymbol{x})$ *where* $\boldsymbol{r}(\boldsymbol{x}) = \square \, range \, (f, \text{pro } \boldsymbol{x})$.

Proof. Denote the relation $f(x) = z$ by $\phi(x, z)$. Therefore $\phi(\text{dual } \boldsymbol{x}, \boldsymbol{z})$ holds because $\boldsymbol{z}$ is $(f, \boldsymbol{x})$-interpretable. Define $\boldsymbol{y} = \square \{(x, z) \in (\text{pro } \boldsymbol{x}, \text{pro } \boldsymbol{z}) | \phi(x, z)\}$, which is well defined thanks to the first assertion of Proposition 4.2 (page 144). Obviously, $\boldsymbol{y} \subseteq$ (pro $\boldsymbol{x}, \boldsymbol{r}(\boldsymbol{x})$). Now, by the third assertion of Proposition 4.2, we have $\phi(\text{dual } \boldsymbol{x}, \boldsymbol{z}) \Longrightarrow \phi(\boldsymbol{y} \wedge (\text{dual } \boldsymbol{x}, \boldsymbol{z}))$, and by the Proposition 4.1 (page 143), $\phi(\boldsymbol{y} \wedge (\text{dual } \boldsymbol{x}, \boldsymbol{z})) \Longrightarrow \phi((\text{pro } \boldsymbol{x}, \boldsymbol{r}(\boldsymbol{x})) \wedge (\text{dual } \boldsymbol{x}, \boldsymbol{z}))$. Now, because dual $\boldsymbol{x} \subseteq$ pro $\boldsymbol{x}$, we have (dual $\boldsymbol{x}$) $\wedge$ (pro $\boldsymbol{x}$) = dual $\boldsymbol{x}$ and therefore $\phi(\text{dual } \boldsymbol{x}, \boldsymbol{z} \wedge \boldsymbol{r}(\boldsymbol{x}))$, i.e. $\boldsymbol{z} \wedge \boldsymbol{r}(\boldsymbol{x})$ is $(f, \boldsymbol{x})$-interpretable.

Furthermore, $\boldsymbol{z} \wedge \boldsymbol{r}(\boldsymbol{x}) \subseteq \boldsymbol{z}$ so by the minimality of $\boldsymbol{z}$, we have $\boldsymbol{z} \wedge \boldsymbol{r}(\boldsymbol{x}) = \boldsymbol{z}$. This eventually implies $\boldsymbol{z} \subseteq \boldsymbol{r}(\boldsymbol{x})$. $\square$

Proposition 6.6. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a locally Lipschitz continuous function and $\boldsymbol{f}$ be one minimal AE-extension of f. Then for all $\boldsymbol{x}^{\mathsf{ref}} \in \mathbb{IR}^n$, there exists $\gamma \in \mathbb{R}$, $\gamma > 0$, such that for any $\boldsymbol{x} \in \boldsymbol{K}\boldsymbol{x}^{\mathsf{ref}}$,*

$$|| \operatorname{wid} \boldsymbol{f}(\boldsymbol{x})|| \leq \gamma || \operatorname{wid} \boldsymbol{x}||.$$

Proof. By Proposition 5.3 (page 150) and Lemma 6.2 we have

$$\operatorname{imp} \boldsymbol{r}(\boldsymbol{x}) \subseteq \boldsymbol{f}(\boldsymbol{x}) \subseteq \boldsymbol{r}(\boldsymbol{x}),$$

where $\boldsymbol{r}(\boldsymbol{x}) = \square \operatorname{range}(f, \operatorname{pro} \boldsymbol{x})$. These two inclusion obviously imply

$$|| \operatorname{rad} \boldsymbol{f}(\boldsymbol{x})|| \leq || \operatorname{rad} \boldsymbol{r}(\boldsymbol{x})||. \tag{5}$$

Now, it is proved in [14] that if f is locally Lipschitz continuous then so is $\boldsymbol{r}(\boldsymbol{x})$ (this is a particular case of Theorem 2.6 proved there). Therefore, for all $\boldsymbol{x}^{\mathsf{ref}} \in \mathbb{IR}^n$, there exists a $\gamma \in \mathbb{R}$, $\gamma > 0$ such that for any $\boldsymbol{x} \in \boldsymbol{K}\boldsymbol{x}^{\mathsf{ref}}$ and any $\boldsymbol{y} \in \boldsymbol{K}\boldsymbol{x}^{\mathsf{ref}}$,

$$\operatorname{dist}(\boldsymbol{r}(\boldsymbol{x}), \boldsymbol{r}(\boldsymbol{y})) \leq \gamma \operatorname{dist}(\boldsymbol{x}, \boldsymbol{y}).$$

Choose $\boldsymbol{y} = \operatorname{mid} \boldsymbol{x} \in \boldsymbol{K}\boldsymbol{x}^{\mathsf{ref}}$ and notice that $\operatorname{dist}(\boldsymbol{x}, \operatorname{mid} \boldsymbol{x}) = || \operatorname{rad} \boldsymbol{x}||$ so as to obtain $\operatorname{dist}(\boldsymbol{r}(\boldsymbol{x}), f(\operatorname{mid} \boldsymbol{x})) \leq \gamma || \operatorname{rad} \boldsymbol{x}||$. Finally, notice that $|| \operatorname{rad} \boldsymbol{r}(\boldsymbol{x})|| \leq \operatorname{dist}(\boldsymbol{r}(\boldsymbol{x}), z)$ obviously holds for any $z \in \mathbb{R}^n$ and hence in particular for $z = f(\operatorname{mid} \boldsymbol{x})$. Therefore, thanks to (5), we have proved $|| \operatorname{rad} \boldsymbol{f}(\boldsymbol{x})|| \leq \gamma || \operatorname{rad} \boldsymbol{x}||$, which is equivalent to the statement of the proposition. $\square$

7 Order of Convergence of AE-Extensions

In the context of classical interval extensions, the quality of an extension is characterized through its order of convergence. This notion is carried to AE-extensions.

Definition 7.1. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous function and $\boldsymbol{g} : \mathbb{KR}^n \longrightarrow \mathbb{KR}^m$ be an AE-extension of f. The AE-extension $\boldsymbol{g}$ has a convergence order $\alpha \in \mathbb{R}$, $\alpha > 0$, if and only if there exists a minimal AE-extension $\boldsymbol{f}$ of f more accurate that $\boldsymbol{g}$ such that for any $\boldsymbol{x}^{\mathsf{ref}} \in \mathbb{IR}^n$, there exists $\gamma > 0$ such that for any $\boldsymbol{x} \in \boldsymbol{K}\boldsymbol{x}^{\mathsf{ref}}$,*

$$|| \operatorname{wid} \boldsymbol{g}(\boldsymbol{x}) - \operatorname{wid} \boldsymbol{f}(\boldsymbol{x})|| \leq \gamma (|| \operatorname{wid} \boldsymbol{x}||)^\alpha.$$

Remark 11. It is obvious that an AE-extension which has an order of convergence α has also an order of convergence α' for any $0 < \alpha' \leq \alpha$. Also, the usually considered orders of convergence are integers. An order of convergence 1 is called a linear convergence, and an order of convergence 2 a quadratic order of convergence.

Remark 12. For any AE-extension $\boldsymbol{g}$ of f, the existence of at least one minimal AE-extension of f which is more accurate than $\boldsymbol{g}$ was established by Proposition 6.3. Therefore, the statement of the previous definition cannot fail due to a lack of minimal AE-extensions. Also, if the existence of an unique minimal AE-extension is assumed, the definition of the order of convergence of AE-extensions coincides with its usual definition in the context of classical interval theory.

In the context of classical interval extensions, the order of convergence is related to the distance to the minimal extension. So as to extend this property to the AE-extensions, the following lemma is first established:

Lemma 7.1. *Let $z \in \mathbb{KR}^m$ and $z' \in \mathbb{KR}^m$ be such that $z' \subseteq z$. Then,*

$$\frac{1}{2}||\operatorname{wid} z - \operatorname{wid} z'|| \leq \operatorname{dist}(z, z') \leq ||\operatorname{wid} z - \operatorname{wid} z'||.$$

Proof. The inclusion $z' \subseteq z$ implies $0 \subseteq z - \operatorname{dual} z'$ and therefore $u := z - \operatorname{dual} z'$ is proper and contains 0. Then notice that both $||\operatorname{wid} z - \operatorname{wid} z'|| = ||\operatorname{wid}(z - \operatorname{dual} z')|| = ||\operatorname{wid} u||$ and $\operatorname{dist}(z, z') = ||u||$. Finally the following inequality obviously hold for any proper interval that contains 0: $||\operatorname{rad} u|| \leq ||u|| \leq ||\operatorname{wid} u||$. This corresponds to the statement of the lemma. $\qquad\square$

The order of convergence of an AE-extension is now proved to be related to the distance to some minimal AE-extension.

Proposition 7.1. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous function and $g : \mathbb{KR}^n \longrightarrow \mathbb{KR}^m$ be an AE-extension of f. The AE-extension g has a convergence order $\alpha \in \mathbb{R}$, $\alpha > 0$, if and only if there exists an minimal AE-extension f of f more accurate that g such that for any $x^{\mathsf{ref}} \in \mathbb{IR}^n$, there exists $\gamma > 0$ such that for any $x \in Kx^{\mathsf{ref}}$,*

$$\operatorname{dist}(g(x), f(x)) \leq \gamma(||\operatorname{wid} x||)^\alpha.$$

Proof. Suppose that g has an order of convergence α, that is there exists an minimal AE-extension f more accurate than g such that for all x^{ref}, there exists γ, for all $x \in Kx^{\mathsf{ref}}$,
$$||\operatorname{wid} g(x) - \operatorname{wid} f(x)|| \leq \gamma(||\operatorname{wid} x||)^\alpha.$$
Then, by the Lemma 7.1, $\operatorname{dist}(g(x), f(x)) \leq ||\operatorname{wid} g(x) - \operatorname{wid} f(x)||$ which proves that the property stated by the present proposition is true. Now suppose that this property is true, that is there exists an minimal AE-extension f more accurate than g such that for all x^{ref}, there exists γ, for all x such that pro $x \subset x^{\mathsf{ref}}$,

$$\operatorname{dist}(g(x), f(x)) \leq \gamma(||\operatorname{wid} x||)^\alpha.$$

Then, by the Lemma 7.1, $\frac{1}{2}||\operatorname{wid} g(x) - \operatorname{wid} f(x)|| \leq \operatorname{dist}(g(x), f(x))$ which shows that g has an order of convergence α using the definition 7.1 with $\gamma' = 2\gamma$. $\qquad\square$

Finally, next proposition states that any locally Lipschitz continuous AE-extension has a linear convergence. This property generalizes a well known property of classical intervals extensions.

Proposition 7.2. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous function and $g : \mathbb{KR}^n \longrightarrow \mathbb{KR}^m$ be an AE-extension of f. If g is locally Lipschitz continuous then it has a linear convergence.*

Proof. First of all, it is easy to check that if g is locally Lipschitz continuous, then so is f (because $f(x) = g(x)$ for all $x \in \mathbb{R}^n$). Then pick up a minimal AE-extension f more accurate than g (which exists thanks to Proposition 6.3). We use the local Lipschitz continuity of g and Proposition 6.6 (page 155) so as to prove that for all $x^{\mathsf{ref}} \in \mathbb{IR}^n$, there exists $\gamma > 0$ and $\gamma' > 0$ such that for all $x \in Kx^{\mathsf{ref}}$ and all $y \in Kx^{\mathsf{ref}}$,

$$||\operatorname{wid} f(x)|| \leq \gamma||\operatorname{wid} x|| \quad \text{and} \quad \operatorname{dist}(g(x), g(y)) \leq \gamma' \operatorname{dist}(x, y).$$

Then choosing $\boldsymbol{y} = [\text{mid } \boldsymbol{x}, \text{mid } \boldsymbol{x}] \subseteq \boldsymbol{x}^{\text{ref}}$, we obtain

$$\text{dist}(\boldsymbol{g}(\boldsymbol{x}), \boldsymbol{g}(\text{mid } \boldsymbol{x})) \leq \gamma' || \text{rad } \boldsymbol{x}||$$

We have $|| \text{rad } \boldsymbol{g}(\boldsymbol{x})|| \leq \text{dist}(\boldsymbol{g}(\boldsymbol{x}), \boldsymbol{g}(\text{mid } \boldsymbol{x}))$ (because $\boldsymbol{g}(\text{mid } \boldsymbol{x})$ is degenerate) and therefore $|| \text{rad } \boldsymbol{g}(\boldsymbol{x})|| \leq \gamma' || \text{rad } \boldsymbol{x}||$. Now define $\gamma'' = \max\{\gamma, \gamma'\}$. Therefore, we have proved that

$$|| \text{wid } \boldsymbol{f}(\boldsymbol{x})|| \leq \gamma'' || \text{wid } \boldsymbol{x}|| \quad \text{and} \quad || \text{wid } \boldsymbol{g}(\boldsymbol{x})|| \leq \gamma'' || \text{wid } \boldsymbol{x}||.$$

So, we have $|| \text{wid } \boldsymbol{g}(\boldsymbol{x}) - \text{wid } \boldsymbol{f}(\boldsymbol{x})|| \leq 2\gamma'' || \text{wid } \boldsymbol{x}||$ (the worth case happening when $\text{wid } \boldsymbol{g}(\boldsymbol{x}) = - \text{wid } \boldsymbol{f}(\boldsymbol{x})$). $\square$

8   Outward Rounding for AE-Extensions

When using computers, one has to deal with the finiteness of the real numbers representation. This leads to incorrect computations. In the context of classical intervals theory, outward rounding is compatible with semantic of the interval extensions and therefore allows conducting reliable computations using a finite precision for numbers representation. In the context of AE-extensions, it is now proved that outward rounding can be used in a similar way. Similarly to the classical interval extensions, the outward rounding of AE-extensions is proceeded in two steps:

1. outward rounding of the interval arguments
2. outward rounding of the AE-extension

The outward rounding of an interval $\boldsymbol{x} \in \mathbb{KR}^n$ is denoted by $\Diamond\boldsymbol{x}$ and satisfies by definition $\boldsymbol{x} \subseteq \Diamond\boldsymbol{x}$. On the one hand, the outward rounding of a proper interval enlarges the involved set of reals: for example $\Diamond[-1, 1]$ could be equal to $[-1.1, 1.1]$. On the other hand, the outward rounding of an improper interval retracts the involved set of reals: for example $\Diamond[1, -1]$ could be equal to $[0.9, -0.9]$. It can also happen that the outward rounding changes the proper/improper quality of an interval: for example $\Diamond[0.05, -0.05]$ could be equal to $[-0.05, 0.05]$. The outward rounding of an AE-extension $\boldsymbol{g}$ is denoted by $\boldsymbol{g}^{\Diamond}$ and satisfies by definition $\boldsymbol{g}(\boldsymbol{x}) \subseteq \boldsymbol{g}^{\Diamond}(\boldsymbol{x})$ for any $\boldsymbol{x} \in \mathbb{KR}^n$. The next proposition proves that this outward rounding process is compatible with the interpretations of the AE-extensions.

Proposition 8.1. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous function and $\boldsymbol{g} : \mathbb{KR}^n \longrightarrow \mathbb{KR}^m$ be an AE-extension of f. Then, the interval function $\boldsymbol{h} : \mathbb{KR}^n \longrightarrow \mathbb{KR}^m$ defined by $\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{g}^{\Diamond}(\Diamond\boldsymbol{x})$ is a weak AE-extension of f.*

Proof. Denote the relation $f(x) = z$ by $\phi(x, z)$. Then, as $\boldsymbol{g}$ is an AE-extension of f, the interval $\boldsymbol{g}(\Diamond\boldsymbol{x})$ is $(f, \Diamond\boldsymbol{x})$-interpretable for all $\boldsymbol{x} \in \mathbb{KR}^n$, that is

$$\phi(\text{dual } \Diamond\boldsymbol{x}, \boldsymbol{g}(\Diamond\boldsymbol{x}))$$

is true. As $\text{dual } \Diamond\boldsymbol{x} \subseteq \text{dual } \boldsymbol{x}$ and $\boldsymbol{g}(\Diamond\boldsymbol{x}) \subseteq \boldsymbol{g}^{\Diamond}(\Diamond\boldsymbol{x})$ we can apply Proposition 4.1 which proves that

$$\phi(\text{dual } \boldsymbol{x}, \boldsymbol{g}^{\Diamond}(\Diamond\boldsymbol{x}))$$

is true (for example, $\forall x \in \Diamond\boldsymbol{x}, \exists z \in \boldsymbol{g}(\Diamond\boldsymbol{x}), z = f(x)$ implies $\forall x \in \boldsymbol{x}, \exists z \in \Diamond\boldsymbol{g}(\Diamond\boldsymbol{x}), z = f(x)$). Therefore, $\boldsymbol{h}(\boldsymbol{x})$ is $(f, \boldsymbol{x})$-interpretable. $\square$

The weak AE-extension $\boldsymbol{g}^{\Diamond}(\Diamond\boldsymbol{x})$ involves only representable numbers, and hence can be used in the context of a finite representation of real numbers. The next example illustrates how the minimal AE-extension of ln can be rounded.

Example 8.1. One wants to compute some inner and outer approximations of

$$\text{range}\,(\ln, \boldsymbol{x}) = [1, 2]$$

with $\boldsymbol{x} = [e, e^2]$ using a two decimals precision. Consider first the outer approximation: to compute an outer approximation of the range, a rounded approximation of $\ln^*(\boldsymbol{x})$ has to be computed. First $\boldsymbol{x}$ is rounded to $\Diamond[e, e^2] = [2.71, 7.39]$. Then, the rounded AE-extension gives

$$\Diamond\ln^*(\Diamond\boldsymbol{x}) = \Diamond[0.99694863... \,,\, 2.0001277...] = [0.99, 2.01],$$

which is indeed an outer approximation of the range. Now consider the inner approximation: so as to compute an inner approximation of the range, a rounded approximation of $\ln^*(\text{dual } \boldsymbol{x})$ has to be computed. First, dual $\boldsymbol{x} = [e^2, e]$ is rounded to $\Diamond[e^2, e] = [7.38, 2.72]$. Notice that the outward rounding of an improper interval computes an inner rounding on its proper projection: the underlying set of reals has been retracted by the outward rounding. Then, the rounded AE-extension leads to

$$\Diamond\ln^*(\Diamond(\text{dual } \boldsymbol{x})) = \Diamond[1.9987736... \,,\, 1.0006319...] = [1.99, 1.01].$$

Once more, the outward rounding of an improper interval computes an inner rounding of its proper projection, and the final result is indeed an inner approximation of the range.

Finally, the AE-extensions which will be met in the sequel will be composed of several interval functions. Suppose that an AE-extension $\boldsymbol{g}$ is obtained through the composition of some interval functions $\boldsymbol{g}_k$. Suppose that these interval functions are increasing with respect to the inclusion. Then, the interval function obtained through the composition of the outward rounded interval functions $\boldsymbol{g}_k^{\Diamond}$ is obviously an outward rounding of $\boldsymbol{g}$. Therefore, rounding compound AE-extensions can be done rounding the interval functions met in its composition.

Now, an outward rounded AE-extension can also be obtained if the expression of $\boldsymbol{g}$ contains non-increasing interval functions provided that their arguments are leaves of the $\boldsymbol{g}$ expression. For example, consider $\boldsymbol{g}(\boldsymbol{x}) = (\text{pro } \boldsymbol{x}) + (\text{pro } \boldsymbol{x})$. Although the interval function pro is not increasing, the interval function $\boldsymbol{g}^{\Diamond}(\boldsymbol{x}) = (\text{pro}^{\Diamond}\boldsymbol{x}) +^{\Diamond} (\text{pro}^{\Diamond}\boldsymbol{x})$ is an outward rounding of the function $\boldsymbol{g}$. Indeed, $(\text{pro}^{\Diamond}\boldsymbol{x}) \supseteq (\text{pro } \boldsymbol{x})$ because $\text{pro}^{\Diamond}\boldsymbol{x}$ is an outward rounding of pro . As $+$ is increasing,

$$(\text{pro } \boldsymbol{x}) + (\text{pro } \boldsymbol{x}) \subseteq (\text{pro}^{\Diamond}\boldsymbol{x}) + (\text{pro}^{\Diamond}\boldsymbol{x}),$$

and because $+^{\Diamond}$ is an outward rounding of $+$,

$$(\text{pro}^{\Diamond}\boldsymbol{x}) + (\text{pro}^{\Diamond}\boldsymbol{x}) \subseteq (\text{pro}^{\Diamond}\boldsymbol{x}) +^{\Diamond} (\text{pro}^{\Diamond}\boldsymbol{x}).$$

All AE-extensions proposed in this paper fulfill this hypothesis and thus can easily be rounded.

9 The Minimal AE-Extension f^* of Real-Valued Functions

The special case of real valued functions, i.e. $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, is now investigated. The next theorem proves that the lattice operations of generalized intervals gives rise to a useful expression of the unique minimal AE-extension of a real-valued function.

Theorem 9.1. *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *be a continuous function and* $\boldsymbol{x} \in \mathbb{KR}^n$. *Then,* $\boldsymbol{z} \in \mathbb{KR}$ *is* $(f, \boldsymbol{x})$-*interpretable if and only if*

$$\bigvee_{x_{\mathcal{P}} \in \boldsymbol{x}_{\mathcal{P}}} \bigwedge_{x_{\mathcal{I}} \in (\mathrm{pro}\ \boldsymbol{x}_{\mathcal{I}})} f(x) \ \subseteq \ \boldsymbol{z}, \tag{6}$$

where $\mathcal{P} = \mathcal{P}(\boldsymbol{x})$ *and* $\mathcal{I} = \mathcal{I}(\boldsymbol{x})$ *(if one of these sets of indices is empty, the corresponding operation is simply canceled in* (6)*).*

Proof. We have to prove that whatever is $\boldsymbol{z} \in \mathbb{KR}$, the following quantified proposition

$$\big(\forall x_{\mathcal{P}} \in \boldsymbol{x}_{\mathcal{P}}\big)\big(\boldsymbol{q}^{(\boldsymbol{z})}z \in \mathrm{pro}\ \boldsymbol{z}\big)\big(\exists x_{\mathcal{I}} \in \mathrm{pro}\ \boldsymbol{x}_{\mathcal{I}}\big)\big(f(x) = z\big) \tag{7}$$

is true if and only if (6) is true, the quantifier $\boldsymbol{q}^{(\boldsymbol{z})}$ being defined like in Corollary 3.1. We suppose that both $\mathcal{P}$ and $\mathcal{I}$ are not empty, the other cases being similar and simpler. For any values of $x_{\mathcal{P}} \in \boldsymbol{x}_{\mathcal{P}}$, the set $\big\{z \in \mathbb{R} | \big(\exists x_{\mathcal{I}} \in \boldsymbol{x}_{\mathcal{I}}\big)\big(f(x) = z\big)\big\}$ is an interval because f is continuous. This interval is denoted by $\boldsymbol{r}(x_{\mathcal{P}}, \boldsymbol{x}_{\mathcal{I}})$ and we have both

$$\boldsymbol{r}(x_{\mathcal{P}}, \boldsymbol{x}_{\mathcal{I}}) = \bigvee_{x_{\mathcal{I}} \in (\mathrm{pro}\ \boldsymbol{x}_{\mathcal{I}})} f(x) \quad \text{and} \quad \mathrm{dual}\ \boldsymbol{r}(x_{\mathcal{P}}, \boldsymbol{x}_{\mathcal{I}}) = \bigwedge_{x_{\mathcal{I}} \in (\mathrm{pro}\ \boldsymbol{x}_{\mathcal{I}})} f(x).$$

Now, $\big(\exists x_{\mathcal{I}} \in \mathrm{pro}\ \boldsymbol{x}_{\mathcal{I}}\big)\big(f(x) = z\big)$ is equivalent to $z \in \boldsymbol{r}(x_{\mathcal{P}}, \boldsymbol{x}_{\mathcal{I}})$. So the quantified proposition (7) is equivalent to

$$\big(\forall x_{\mathcal{P}} \in \boldsymbol{x}_{\mathcal{P}}\big)\big(\boldsymbol{q}^{(\boldsymbol{z})}z \in \mathrm{pro}\ \boldsymbol{z}\big)\big(z \in \boldsymbol{r}(x_{\mathcal{P}}, \boldsymbol{x}_{\mathcal{I}})\big).$$

Applying the corollary 3.1, we have

$$\big(\boldsymbol{q}^{(\boldsymbol{z})}z \in \mathrm{pro}\ \boldsymbol{z}\big)\big(z \in \boldsymbol{r}(x_{\mathcal{P}}, \boldsymbol{x}_{\mathcal{I}})\big) \iff \boldsymbol{z} \supseteq (\mathrm{dual}\ \boldsymbol{r}(x_{\mathcal{P}}, \boldsymbol{x}_{\mathcal{I}})).$$

Therefore, the quantified proposition (7) is equivalent to

$$\big(\forall x_{\mathcal{P}} \in \boldsymbol{x}_{\mathcal{P}}\big)\big(\boldsymbol{z} \supseteq (\mathrm{dual}\ \boldsymbol{r}(x_{\mathcal{P}}, \boldsymbol{x}_{\mathcal{I}}))\big).$$

Finally, by the definition of the least upper bound, this is equivalent to

$$\boldsymbol{z} \supseteq \bigvee_{x_{\mathcal{P}} \in \boldsymbol{x}_{\mathcal{P}}} (\mathrm{dual}\ \boldsymbol{r}(x_{\mathcal{P}}, \boldsymbol{x}_{\mathcal{I}})),$$

which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

As direct consequence of Proposition 6.4 (page 154), the interval

$$\bigvee_{x_{\mathcal{P}} \in \boldsymbol{x}_{\mathcal{P}}} \bigwedge_{x_{\mathcal{I}} \in (\mathrm{pro}\ \boldsymbol{x}_{\mathcal{I}})} f(x)$$

is the unique minimal $(f, \boldsymbol{x})$-interpretable interval. The same notation f^* as in [34] is kept:

Definition 9.1. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function. The generalized interval function $f^* : \mathbb{KR}^n \longrightarrow \mathbb{KR}$ is defined in the following way:

$$
\begin{aligned}
f^*(\boldsymbol{x}) &= \bigvee_{x_{\mathcal{P}} \in \boldsymbol{x}_{\mathcal{P}}} \bigwedge_{x_{\mathcal{I}} \in (\mathrm{pro}\ \boldsymbol{x}_{\mathcal{I}})} f(x) \\
&= [\min_{x_{\mathcal{P}} \in \boldsymbol{x}_{\mathcal{P}}} \max_{x_{\mathcal{I}} \in (\mathrm{pro}\ \boldsymbol{x}_{\mathcal{I}})} f(x)\ ,\ \max_{x_{\mathcal{P}} \in \boldsymbol{x}_{\mathcal{P}}} \min_{x_{\mathcal{I}} \in (\mathrm{pro}\ \boldsymbol{x}_{\mathcal{I}})} f(x)],
\end{aligned}
$$

where $\mathcal{P} = \mathcal{P}(\boldsymbol{x})$ et $\mathcal{I} = \mathcal{I}(\boldsymbol{x})$.

Remark 13. When $\mathcal{P} = \emptyset$ or $\mathcal{I} = \emptyset$, the expressions of f^* are respectively

$$
f^*(\boldsymbol{x}) = [\max_{x \in (\mathrm{pro}\ \boldsymbol{x})} f(x),\ \min_{x \in (\mathrm{pro}\ \boldsymbol{x})} f(x)]\ \text{ and }\ f^*(\boldsymbol{x}) = [\min_{x \in \boldsymbol{x}} f(x), \max_{x \in \boldsymbol{x}} f(x)].
$$

However, the use of the general expressions of Definition 9.1 in all cases is allowed in the sequel.

The next three examples illustrate some computations and some interpretations of f^*.

Example 9.1. Consider the function $f(x, y) = x^2 + y^2$ and the proper intervals $\boldsymbol{x} = [-3, 3]$ et $\boldsymbol{y} = [4, 6]$. Then,

$$
\begin{aligned}
f^*(\boldsymbol{x}, \boldsymbol{y}) &= [\min_{x \in \boldsymbol{x}} \min_{y \in \boldsymbol{y}} f(x, y)\ ,\ \max_{x \in \boldsymbol{x}} \max_{y \in \boldsymbol{y}} f(x, y)] \\
&= [\min_{x \in \boldsymbol{x}} x^2 + \underline{y}^2\ ,\ \max_{x \in \boldsymbol{x}} x^2 + \overline{y}^2] \\
&= [0 + \underline{y}^2\ ,\ \overline{x}^2 + \overline{y}^2] \\
&= [16\ ,\ 9 + 36] = [16, 45].
\end{aligned}
$$

As f^* is the unique minimal AE-extension of f, the previous computation proves that $\boldsymbol{z} = [16, 45]$ is the smallest interval that satisfies

$$
(\forall x \in \boldsymbol{x})(\forall y \in \boldsymbol{y})(\exists z \in \boldsymbol{z})(x^2 + y^2 = z).
$$

Example 9.2. In the same situation as in the previous example, compute now

$$
\begin{aligned}
f^*(\mathrm{dual}\ \boldsymbol{x}, \mathrm{dual}\ \boldsymbol{y}) &= [\max_{x \in \boldsymbol{x}} \max_{y \in \boldsymbol{y}} f(x, y)\ ,\ \min_{x \in \boldsymbol{x}} \min_{y \in \boldsymbol{y}} f(x, y)] \\
&= [45, 16].
\end{aligned}
$$

As f^* is the unique minimal AE-extension of f, the previous computation proves that $\boldsymbol{z} = \mathrm{pro}\ [45, 16] = [16, 45]$ is the largest interval that satisfies

$$
(\forall z \in \boldsymbol{z})(\exists x \in \boldsymbol{x})(\exists y \in \boldsymbol{y})(x^2 + y^2 = z).
$$

Example 9.3. In the same situation as the two previous examples, now compute

$$
\begin{aligned}
f^*(\boldsymbol{x}, \mathrm{dual}\ \boldsymbol{y}) &= [\min_{x \in \boldsymbol{x}} \max_{y \in \boldsymbol{y}} f(x, y)\ ,\ \max_{x \in \boldsymbol{x}} \min_{y \in \boldsymbol{y}} f(x, y)] \\
&= [\min_{x \in \boldsymbol{x}} x^2 + \overline{y}^2\ ,\ \max_{x \in \boldsymbol{x}} x^2 + \underline{y}^2] \\
&= [0 + \overline{y}^2\ ,\ \overline{x}^2 + \underline{y}^2] \\
&= [36\ ,\ 9 + 16] = [36, 25].
\end{aligned}
$$

As f^* is the unique minimal AE-extension of f, the previous computation proves that $\boldsymbol{z} = \mathrm{pro}\ [36, 25] = [25, 36]$ is the largest interval that satisfies

$$
(\forall z \in \boldsymbol{z})(\forall x \in \boldsymbol{x})(\exists y \in \boldsymbol{y})(x^2 + y^2 = z)
$$

Computing f^* is NP-hard with respect to the number of variables, as the exact computation of the range of a function over a box is a special case of computing f^* and is NP-hard (see [20]). As in the context of the classical interval extensions, approximations of f^* will be constructed in the following way:

1. f^* is computed for a set of elementary functions, leading to a generalized interval arithmetic. This arithmetic will be proved to coincide with Kaucher arithmetic (Section 10).

2. Some natural AE-extensions (generalized interval evaluations of some expression) are defined using generalized interval arithmetic (Section 11).

Before going to these developments, an important property of the AE-extension f^* is shown. First, notice that in the case of a real function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ the minimal outer approximation and the maximal inner approximation of its range are equal. In terms of AE-extensions, this property is written $\boldsymbol{x} \in \mathbb{IR}^n$ implies $f^*(\text{dual } \boldsymbol{x}) = \text{dual } f^*(\boldsymbol{x})$. A natural question that arises is the generalization of this property to arbitrary generalized intervals. The following proposition shows that an inclusion holds in general instead of an equality.

Proposition 9.1. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function and $\boldsymbol{x} \in \mathbb{KR}^n$. Then*

$$f^*(\text{dual } \boldsymbol{x}) \subseteq \text{dual } f^*(\boldsymbol{x}). \tag{8}$$

Proof. By definition of f^*:

$$
\begin{aligned}
f^*(\boldsymbol{x}) &= \bigvee_{x_{\mathcal{P}} \in \boldsymbol{x}_{\mathcal{P}}} \bigwedge_{x_{\mathcal{I}} \in (\text{pro } \boldsymbol{x}_{\mathcal{I}})} f(x) \\
f^*(\text{dual } \boldsymbol{x}) &= \bigvee_{x_{\mathcal{I}} \in (\text{pro } \boldsymbol{x}_{\mathcal{I}})} \bigwedge_{x_{\mathcal{P}} \in \boldsymbol{x}_{\mathcal{P}}} f(x)
\end{aligned}
$$

Since dual reverses the generalized interval lattice operations,

$$\text{dual } f^*(\boldsymbol{x}) = \bigwedge_{x_{\mathcal{P}} \in \boldsymbol{x}_{\mathcal{P}}} \bigvee_{x_{\mathcal{I}} \in (\text{pro } \boldsymbol{x}_{\mathcal{I}})} f(x)$$

The inclusion between the latter two is an unusual property of lattices: Performing intersection before union yields a small element than the reverse order. $\square$

Proposition 9.1 has several consequences. In particular, this implies that at least one of $f^*(\text{dual } \boldsymbol{x})$ or $f^*(\boldsymbol{x})$ has to be improper (both being strictly proper leading to a contradiction). We obtain the following non trivial property: In general for a continuous two variables function and arbitrary bounded intervals $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{IR}$, at least one of the following propositions has to hold:

$$\big(\exists z \in \mathbb{R}\big)\big(\forall x \in \boldsymbol{x}\big)\big(\exists y \in \boldsymbol{y}\big)\big(f(x, y) = z\big)$$

$$\big(\exists z \in \mathbb{R}\big)\big(\forall y \in \boldsymbol{y}\big)\big(\exists x \in \boldsymbol{x}\big)\big(f(x, y) = z\big).$$

The commutation with the dual operator can also be used to study the minimality of AE-extensions. These developments are not presented here (see Proposition IV.18 and Proposition IV.19 page 149 of [7] for more details), the minimality of the natural AE-extension of bilinear function being proved directly in Theorem 12.2 (page 173).

10 AE-Extensions of Elementary Functions

The elementary functions considered here are the following, their definition domain being the usual ones:

- two variables functions: $\Omega = \{\ x+y\ ,\ x-y\ ,\ x \times y\ ,\ x/y\}$
- one variable functions: $\Phi = \{\ \exp x,\ \ln x,\ \sin x,\ \cos x,\ \tan x,\ \arccos x,\ \arcsin x,\ \arctan x,\ \mathrm{abs}\,x,\ x^n,\ \sqrt[n]{x}\}$.

In the cases of these simple functions, the minimal AE-extension f^* can be computed formally, leading to a generalized interval arithmetic. However the explicit computations of these expressions will not be presented: using some properties of Kaucher arithmetic, the expressions of f^* are proved to raise the same results than the expressions of $f^{\mathbb{KR}}$ for these operations.

Also, some properties of the minimal AE-extensions of these elementary functions are stated through Proposition 10.1 and Proposition 10.2. These important properties will be used in the next section in order to investigate the generalized interval evaluation of an expression.

One variable elementary functions

In the case of one variable elementary functions, the definitions of f^* and $f^{\mathbb{KR}}$ coincide:

$$f^*(\boldsymbol{x}) = f^{\mathbb{KR}}(\boldsymbol{x}) = \bigwedge_{x \in \mathrm{pro}\ \boldsymbol{x}}^{\boldsymbol{x}} [f(x), f(x)]$$

That is,

- if $\boldsymbol{x} \in \mathbb{IR}$ then

$$f^*(\boldsymbol{x}) = [\min_{x \in \boldsymbol{x}} f(x), \max_{x \in \boldsymbol{x}} f(x)] = \mathrm{range}\,(f, \boldsymbol{x})\,;$$

- if $\boldsymbol{x} \in \overline{\mathbb{IR}}$ then

$$f^*(\boldsymbol{x}) = [\max_{x \in \mathrm{pro}\ \boldsymbol{x}} f(x), \min_{x \in \mathrm{pro}\ \boldsymbol{x}} f(x)] = \mathrm{dual}\ \mathrm{range}\,(f, \mathrm{pro}\ \boldsymbol{x})\,.$$

In the sequel, both interval extensions f^* and $f^{\mathbb{KR}}$ of the elementary functions of Φ will be denoted by the symbol of the original function.

Example 10.1. Few minimal AE-extension are presented.

- $\exp(\boldsymbol{x}) = [\exp(\underline{x}), \exp(\overline{x})]$
- $\ln(\boldsymbol{x}) = [\ln(\underline{x}), \ln(\overline{x})]$ is defined if both $\underline{x} > 0$ and $\overline{x} > 0$
- $\sqrt{\boldsymbol{x}} = [\sqrt{\underline{x}}, \sqrt{\overline{x}}]$ is defined if both $\underline{x} \geq 0$ and $\overline{x} \geq 0$.

In general, the algorithms dedicated to the computation of the classical interval arithmetic are easily adapted to compute generalized interval arithmetic: only the rounding process has to be adapted.

Now that the AE-extensions of the elementary functions are built, one additional property is needed for the coming developments: each quantified proposition that corresponds to the interpretation of theses minimal AE-extensions must have a **continuous** Skolem function. This is formally stated by the next proposition.

Proposition 10.1. *Let f be an elementary function of Φ and $\boldsymbol{x}_{\{0,1\}} \in \mathbb{KR}^2$ that satisfies $\boldsymbol{x}_0 = f^*(\boldsymbol{x}_1)$. Define the following sets of indices: $\mathcal{A} = \mathcal{P}(\boldsymbol{x}_{\{1\}}) \bigcup \mathcal{I}(\boldsymbol{x}_{\{0\}})$ and $\mathcal{E} = \mathcal{I}(\boldsymbol{x}_{\{1\}}) \bigcup \mathcal{P}(\boldsymbol{x}_{\{0\}})$ (so that $\mathcal{A}$ contains the indices of the universally quantified variables and $\mathcal{E}$ contains the indices of the existentially quantified ones). Then both $\mathcal{A}$ and $\mathcal{E}$ are nonempty (so either $\mathcal{A} = \{0\}$ and $\mathcal{E} = \{1\}$ or $\mathcal{A} = \{1\}$ and $\mathcal{E} = \{0\}$) and the quantified proposition*

$$\big(\forall x_\mathcal{A} \in \mathrm{pro}\ \boldsymbol{x}_\mathcal{A}\big)\big(\exists x_\mathcal{E} \in \mathrm{pro}\ \boldsymbol{x}_\mathcal{E}\big)\big(x_0 = f(x_1)\big)$$

has a **continuous** *Skolem function.*

Proof. Provided in Appendix C page 178. $\qquad\square$

Example 10.2. Consider $f(x) = x^2$ and $\boldsymbol{x} = [2, -1]$ and $\boldsymbol{z} = \boldsymbol{x}^2 = [4, 0]$. So the following quantified proposition is true:

$$\big(\forall z \in \mathrm{pro}\ \boldsymbol{z}\big)\big(\exists x \in \mathrm{pro}\ \boldsymbol{x}\big)\big(f(x) = z\big),$$

that is pro $\boldsymbol{z} \subseteq \mathrm{range}\,(f, \mathrm{pro}\ \boldsymbol{x})$. Proposition 10.1 proves that there exists a **continuous** function $s : \mathrm{pro}\ \boldsymbol{z} \longrightarrow \mathrm{pro}\ \boldsymbol{x}$ which satisfies $x = s(z) \Longrightarrow z = f(x)$. This function is $s(z) = \sqrt{x}$ and indeed range $(s, \mathrm{pro}\ \boldsymbol{z}) \subseteq \mathrm{pro}\ \boldsymbol{x}$ and $x = \sqrt{z} \Longrightarrow x^2 = z$.

Remark 14. There exist continuous one variable real functions which does not satisfy Proposition 10.1 (e.g. consider the function $f(x) = x^3 - x$ and the interval $\boldsymbol{x} = [1, -1]$). Therefore, before adding a new elementary function to Φ, one has to check that it satisfies Proposition 10.1.

Two variable elementary functions

In the case of two variables elementary functions, the expressions of $f^*(\boldsymbol{x}, \boldsymbol{y})$ and $f^{\mathbb{KR}}(\boldsymbol{x}, \boldsymbol{y})$ coincide in the following cases:

- $\boldsymbol{x} \in \mathbb{IR}$ and $\boldsymbol{y} \in \mathbb{IR}$
- $\boldsymbol{x} \in \overline{\mathbb{IR}}$ and $\boldsymbol{y} \in \overline{\mathbb{IR}}$
- $\boldsymbol{x} \in \mathbb{IR}$ and $\boldsymbol{y} \in \overline{\mathbb{IR}}$.

It remains to investigate the case where $\boldsymbol{x} \in \overline{\mathbb{IR}}$ and $\boldsymbol{y} \in \mathbb{IR}$. In this case, the expressions of f^* and $f^{\mathbb{KR}}$ are

$$f^*(\boldsymbol{x}, \boldsymbol{y}) = \vee_{x \in \boldsymbol{x}}\ \wedge_{y \in (\mathrm{pro}\ \boldsymbol{y})}\ f(x,y)\ \ ;\ \ f^{\mathbb{KR}}(\boldsymbol{x}, \boldsymbol{y}) = \wedge_{y \in (\mathrm{pro}\ \boldsymbol{y})}\ \vee_{x \in \boldsymbol{x}}\ f(x,y).$$

These two expressions lead to different results in general. However, it is now proved that they are equal in the cases of the elementary functions of Ω. Given an operation $\circ \in \Omega$, the operation of Kaucher arithmetic is denoted by $\circ^{\mathbb{KR}}$ and the minimal AE-extension by $\circ^*$. First consider a function $\circ \in \{+, \times\}$. If $\boldsymbol{x} \in \overline{\mathbb{IR}}$ and $\boldsymbol{y} \in \mathbb{IR}$, then by the definitions of f^* and $f^{\mathbb{KR}}$, the expression of $\boldsymbol{x} \circ^* \boldsymbol{y}$ coincides with the expression of $\boldsymbol{y} \circ^{\mathbb{KR}} \boldsymbol{x}$. Finally, because $\circ^{\mathbb{KR}}$ is commutative, we have

$$\boldsymbol{x} \circ^* \boldsymbol{y} = \boldsymbol{y} \circ^{\mathbb{KR}} \boldsymbol{x} = \boldsymbol{x} \circ^{\mathbb{KR}} \boldsymbol{y}.$$

In the case of the subtraction, we have $\boldsymbol{x} -^{\mathbb{KR}} \boldsymbol{y} = (-\boldsymbol{y}) +^{\mathbb{KR}} \boldsymbol{x}$ which is proved to be equal to $\boldsymbol{x} -^* \boldsymbol{y}$ in the following way:

$$
\begin{aligned}
(-\boldsymbol{y}) +^{\mathbb{KR}} \boldsymbol{x} &= \vee_{v \in (-\boldsymbol{y})}\ \wedge_{x \in (\mathrm{pro}\ \boldsymbol{x})}\ v + x \\
&= \vee_{y \in \boldsymbol{y}}\ \wedge_{x \in (\mathrm{pro}\ \boldsymbol{x})}\ (-y) + x\ =\ \boldsymbol{x} -^* \boldsymbol{y}.
\end{aligned}
$$

Finally, in the case of the division, we have $\boldsymbol{x}/^{\mathbb{KR}}\boldsymbol{y} = (1/\boldsymbol{y}) \times^{\mathbb{KR}} \boldsymbol{x}$ which is proved to be equal to $\boldsymbol{x}/^{*}\boldsymbol{y}$ in the same way that previously. Therefore, the AE-extensions of the elementary functions of Ω coincide with their $\mathbb{KR}$-extensions. Their expressions are given in Section 2. In the sequel, both interval extensions $\circ^{*}$ and $\circ^{\mathbb{KR}}$ of the elementary functions of Ω will be denoted by the symbol of the original function $\circ$.

Like in the cases of one variable elementary functions, each quantified proposition that corresponds to the interpretation of theses minimal AE-extensions must have a **continuous** Skolem function. This is formally stated by the next proposition.

Proposition 10.2. *Let $\circ \in \Omega$ and $\boldsymbol{x}_{\{0,1,2\}} \in \mathbb{KR}^3$ such that $\boldsymbol{x}_0 = \boldsymbol{x}_1 \circ \boldsymbol{x}_2$. Define the following sets of indices: $\mathcal{A} = \mathcal{P}(\boldsymbol{x}_{\{1,2\}}) \bigcup \mathcal{I}(\boldsymbol{x}_{\{0\}})$ and $\mathcal{E} = \mathcal{I}(\boldsymbol{x}_{\{1,2\}}) \bigcup \mathcal{P}(\boldsymbol{x}_{\{0\}})$ (so that $\mathcal{A}$ contains the indices of the universally quantified variables and $\mathcal{E}$ contains the indices of the existentially quantified ones). Then both $\mathcal{A}$ and $\mathcal{E}$ are nonempty and the quantified proposition*

$$\big(\forall x_{\mathcal{A}} \in \text{pro } \boldsymbol{x}_{\mathcal{A}}\big)\big(\exists x_{\mathcal{E}} \in \text{pro } \boldsymbol{x}_{\mathcal{E}}\big)\big(x_0 = x_1 \circ x_2\big)$$

*has a **continuous** Skolem function.*

Proof. Provided in Appendix C page 178.   □

Example 10.3. Consider $f(x,y) = x + y$ and $\boldsymbol{x} = [1, -1]$, $\boldsymbol{y} = [0, 1]$ and $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y} = [1, 0]$. The following quantified proposition is therefore true:

$$\big(\forall x \in \text{pro } \boldsymbol{x}\big)\big(\forall z \in \text{pro } \boldsymbol{z}\big)\big(\exists y \in \boldsymbol{y}\big)\big(x + y = z\big)$$

The previous proposition provides a stronger statement: it proves that there exists a **continuous** function $s : (\boldsymbol{y}, \text{pro } \boldsymbol{z})^T \longrightarrow \text{pro } \boldsymbol{x}$ which satisfies $x = s(y,z) \implies z = f(x,y)$. In this case, the function s can easily be computed: $s(y,z) = z - y$. Indeed range $(s, \boldsymbol{y}, \text{pro } \boldsymbol{z}) = \text{pro } \boldsymbol{z} - \boldsymbol{y} = [-1, 1]$ and $x = z - y \implies z = x + y$.

Remark 15. There exist continuous two variables real functions which do not satisfy the previous proposition. For example, consider the function

$$f(x,y) = 1 - (x - y)^2$$

and the generalized interval $([1, -1], [1, -1])^T$. Therefore, before adding a new elementary function to Φ, one has to check that it satisfies Proposition 10.2 (see Appendix B for a false assertion entailed by the use of some elementary functions that do not satisfy Proposition 10.2).

11 Natural AE-extensions

The natural AE-extensions consist in evaluating the expression of the function (or a closely related expression) for generalized interval arguments using Kaucher arithmetic. The natural AE-extensions are constructed in two steps:

1. In the case of continuous functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ whose expression contains only one occurrence of each variable, the generalized interval evaluation is proved to raise $(f, \boldsymbol{x})$-interpretable intervals (Subsection 11.1).

2. In the case of continuous functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, multiple occurrences of variables are taken into account: the expression of the function has to be modified before its evaluation with generalized interval arguments (Subsection 11.2).

11.1 Generalized Interval Evaluation of an Expression Containing Only One Occurrence of Each Variable

An introducing example is first investigated. Consider the function $f(x) = (x_1 + x_2)(x_3 + x_4)$ and the generalized interval $\boldsymbol{x} = ([-2,2],[1,-1],[-1,1],[2,-2])^T$. The generalized interval evaluation of the expression of f is decomposed in the following way: $\boldsymbol{t}_1 = \boldsymbol{x}_1 + \boldsymbol{x}_2 = [-1,1]$ and $\boldsymbol{t}_2 = \boldsymbol{x}_3 + \boldsymbol{x}_4 = [1,-1]$. Finally $\boldsymbol{z} = \boldsymbol{t}_1 \times \boldsymbol{t}_2 = [0,0]$. From $\boldsymbol{t}_1 = \boldsymbol{x}_1 + \boldsymbol{x}_2$, the following quantified proposition is true:

$$\big(\forall x_1 \in \boldsymbol{x}_1\big)\big(\exists t_1 \in \boldsymbol{t}_1\big)\big(\exists x_2 \in \text{pro } \boldsymbol{x}_2\big)\big(x_1 + x_2 = t_1\big). \tag{9}$$

From $\boldsymbol{t}_2 = \boldsymbol{x}_3 + \boldsymbol{x}_4$, the following quantified proposition is true:

$$\big(\forall x_3 \in \boldsymbol{x}_1\big)\big(\forall t_2 \in \text{pro } \boldsymbol{t}_2\big)\big(\exists x_4 \in \text{pro } \boldsymbol{x}_4\big)\big(x_3 + x_4 = t_2\big). \tag{10}$$

Finally from $\boldsymbol{z} = \boldsymbol{t}_1 \times \boldsymbol{t}_2$ the following quantified proposition is true:

$$\big(\forall t_1 \in \boldsymbol{t}_1\big)\big(\exists t_2 \in \text{pro } \boldsymbol{t}_2\big)\big(t_1 \times t_2 = 0\big). \tag{11}$$

Now, from (10) and (11), the following quantified proposition is entailed:

$$\big(\forall x_3 \in \boldsymbol{x}_1\big)\big(\forall t_1 \in \boldsymbol{t}_1\big)\big(\exists x_4 \in \text{pro } \boldsymbol{x}_4\big)\big(\exists t_2 \in \text{pro } \boldsymbol{t}_2\big) \\ \big(x_3 + x_4 = t_2 \wedge t_1 \times t_2 = 0\big). \tag{12}$$

Finally from (12) and (9), the following quantified proposition is entailed:

$$\big(\forall x_1 \in \boldsymbol{x}_1\big)\big(\forall x_3 \in \boldsymbol{x}_1\big)\big(\exists x_4 \in \text{pro } \boldsymbol{x}_4\big)\big(\exists x_2 \in \text{pro } \boldsymbol{x}_2\big)\big(\exists t_1 \in \boldsymbol{t}_1\big)\big(\exists t_2 \in \text{pro } \boldsymbol{t}_2\big) \\ \big(x_3 + x_4 = t_2 \wedge t_1 \times t_2 = 0 \wedge x_1 + x_2 = t_1\big),$$

that is

$$\big(\forall x_1 \in \boldsymbol{x}_1\big)\big(\forall x_3 \in \boldsymbol{x}_1\big)\big(\exists x_4 \in \text{pro } \boldsymbol{x}_4\big)\big(\exists x_2 \in \text{pro } \boldsymbol{x}_2\big)\big(f(x) = 0\big). \tag{13}$$

Therefore, the interval $\boldsymbol{z} = [0,0]$ is proved to be $(f,\boldsymbol{x})$-interpretable. Now, in addition to the $(f,\boldsymbol{x})$-interpretability of $\boldsymbol{z}$, the quantified proposition (13) must be proved to have a **continuous** Skolem function, i.e. there exists a **continuous** function $s_{\{2,4\}} : \boldsymbol{x}_{\{1,3\}} \longrightarrow \text{pro } \boldsymbol{x}_{\{2,4\}}$ that satisfies $x_{\{2,4\}} = s_{\{2,4\}}(x_{\{1,3\}}) \Longrightarrow f(x) = 0$. In the context of this introducing example, such a function $s_{\{2,4\}}$ is constructed in the following way: applying Proposition 10.2 to $\boldsymbol{t}_1 = \boldsymbol{x}_1 + \boldsymbol{x}_2$, $\boldsymbol{t}_2 = \boldsymbol{x}_3 + \boldsymbol{x}_4$ and $\boldsymbol{z} = \boldsymbol{t}_1 \times \boldsymbol{t}_2$ respectively proves that the quantified propositions (9), (10) and (11) have some **continuous** Skolem functions, i.e. proves the existence of the following **continuous** functions:

$$\begin{array}{llll} s'_{\{t_1,x_2\}} : \boldsymbol{x}_1 \longrightarrow (\boldsymbol{t}_1, \text{pro } \boldsymbol{x}_2)^T & \text{s.t.} & (t_1, x_2) = s'_{\{t_1,x_2\}}(x_1) \Longrightarrow t_1 = x_1 + x_2 \\ s'_{x_4} : (\text{pro } \boldsymbol{t}_2, \boldsymbol{x}_3)^T \longrightarrow \text{pro } \boldsymbol{x}_4 & \text{s.t.} & x_4 = s'_{x_4}(t_2, x_3) \Longrightarrow t_2 = x_3 + x_4 \\ s'_{t_2} : \boldsymbol{t}_1 \longrightarrow \text{pro } \boldsymbol{t}_2 & \text{s.t.} & t_2 = s'_{t_2}(t_1) \Longrightarrow t_1 \times t_2 = 0 \end{array}$$

The function $s_{\{2,4\}} : \boldsymbol{x}_{\{1,3\}} \longrightarrow \text{pro } \boldsymbol{x}_{\{2,4\}}$ is then constructed in the following way:

$$\begin{array}{rcl} s_2(x_{\{1,3\}}) & = & s'_{x_2}(x_1) \\ s_4(x_{\{1,3\}}) & = & s'_{x_4}\big(s'_{t_2}\big(s'_{t_1}(x_1)\big), x_3\big) \end{array} \tag{14}$$

As a consequence of these definitions, $x_{\{2,4\}} = s_{\{2,4\}}(x_{\{1,3\}})$ implies the existence of $t_1 \in \boldsymbol{t}_1$ and $t_2 \in \text{pro } \boldsymbol{t}_2$ such that $x_4 = s'_{x_4}(t_2, x_3)$ and $t_2 = s'_{t_2}(t_1)$ and $(t_1, x_2) =$

$s'_{\{t_1, x_2\}}(x_1)$. Using the properties satisfied by these three Skolem functions, one obtains $t_1 = x_1 + x_2$ and $t_2 = x_3 + x_4$ and $t_1 \times t_2 = 0$, that is $f(x_{[1..4]}) = 0$. The function $s_{\{2,4\}}$ is therefore a **continuous** Skolem function of the quantified proposition (13).

The next proposition generalizes this introducing example. It will play a key role in the construction of AE-extensions.

Proposition 11.1. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function and f an expression of this function involving elementary functions of Ω and Φ where **each variable has only one occurrence**. For any $\boldsymbol{x}_{[1..n]} \in \mathbb{KR}^n$, define $\boldsymbol{x}_0 = \mathsf{f}(\boldsymbol{x}_{[1..n]})$ where the evaluation is done using Kaucher arithmetic. Furthermore define the sets of indices $\mathcal{A} = \mathcal{P}(\boldsymbol{x}_{[1..n]}) \bigcup \mathcal{I}(\boldsymbol{x}_{\{0\}})$ and $\mathcal{E} = \mathcal{I}(\boldsymbol{x}_{[1..n]}) \bigcup \mathcal{A}(\boldsymbol{x}_{\{0\}})$ (so that $\mathcal{A}$ contains the indices of the universally quantified variable and $\mathcal{E}$ contains the indices of the existentially quantified ones). Then both $\mathcal{A}$ and $\mathcal{E}$ are nonempty and the quantified proposition*

$$\left(\forall x_{\mathcal{A}} \in \text{pro } \boldsymbol{x}_{\mathcal{A}}\right)\left(\exists x_{\mathcal{E}} \in \text{pro } \boldsymbol{x}_{\mathcal{E}}\right)\left(f(x_{[1..n]}) = x_0\right)$$

*has a **continuous** Skolem function (and is therefore true).*

Proof. The proof is conducted by induction over the expression f. First, by the propositions 10.1 and 10.2, the present proposition is true for any elementary functions of Ω and Φ. Therefore, it only remains to prove the induction step of the proposition. Consider the two continuous functions $g : \mathbb{R}^m \longrightarrow \mathbb{R}$ and $h : \mathbb{R}^{m'} \longrightarrow \mathbb{R}$ (whose expressions are respectively g and h) that satisfy the proposition. We just have to prove that their composition also satisfies the proposition (the hypothesis that each variable has only one occurrence is implicitly used here: this composition of two functions does not allow any dependences between some variables). Their composition f is defined using a special numbering of the variables in order to consider the temporary variable in a homogeneous way: the function $f : \mathbb{R}^{m+m'-1} \longrightarrow \mathbb{R}$ is defined by

$$f(x_{\boldsymbol{N}}) = \mathsf{g}\left(x_{[1..m-1]}, \mathsf{h}(x_{[m+1..m+m']})\right)$$

where $\boldsymbol{N} = [1..m-1] \bigcup [m+1..m+m']$.

Example. Defining $\mathsf{f}(x_{\{1,3,4\}}) = \mathsf{g}(x_1, \mathsf{h}(x_{\{3,4\}}))$ allows to decompose the evaluation into $x_0 = g(x_{\{1,2\}})$ and $x_2 = h(x_{\{3,4\}})$.

Then consider any $\boldsymbol{x}_{\boldsymbol{N}} \in \mathbb{KR}^{m+m'-1}$, and define $\boldsymbol{x}_0 = \mathsf{f}(\boldsymbol{x}_{\boldsymbol{N}})$ and $\boldsymbol{x}_m = \mathsf{h}(\boldsymbol{x}_{[m+1..m+m']})$ so that $\boldsymbol{x}_0 = \mathsf{g}(\boldsymbol{x}_{[1..m]})$. Also define the sets of indices $\mathcal{A} = \mathcal{P}(\boldsymbol{x}_{\boldsymbol{N}}) \bigcup \mathcal{I}(\boldsymbol{x}_{\{0\}})$ and $\mathcal{E} = \mathcal{I}(\boldsymbol{x}_{\boldsymbol{N}}) \bigcup \mathcal{P}(\boldsymbol{x}_{\{0\}})$. We have to prove on one hand that both $\mathcal{A}$ and $\mathcal{E}$ are nonempty and on the other hand that there exists a **continuous** function

$$s_{\mathcal{E}} : \text{pro } \boldsymbol{x}_{\mathcal{A}} \longrightarrow \text{pro } \boldsymbol{x}_{\mathcal{E}} \quad \text{s.t.} \quad x_{\mathcal{E}} = s_{\mathcal{E}}(x_{\mathcal{A}}) \Longrightarrow x_0 = \mathsf{f}(x_{\boldsymbol{N}}). \tag{15}$$

Applying the induction hypothesis, we get two **continuous** functions $s'_{\mathcal{E}'} : \text{pro } \boldsymbol{x}_{\mathcal{A}'} \longrightarrow \text{pro } \boldsymbol{x}_{\mathcal{E}'}$ and $s''_{\mathcal{E}''} : \text{pro } \boldsymbol{x}_{\mathcal{A}''} \longrightarrow \text{pro } \boldsymbol{x}_{\mathcal{E}''}$ which satisfy respectively

$$x_{\mathcal{E}'} = s'_{\mathcal{E}'}(x_{\mathcal{A}'}) \Longrightarrow x_0 = g(x_{[1..m]})$$

and

$$x_{\mathcal{E}''} = s''_{\mathcal{E}''}(x_{\mathcal{A}''}) \Longrightarrow x_m = h(x_{[m+1..m+m']}),$$

where $\mathcal{A}'$ and $\mathcal{E}'$ (respectively $\mathcal{A}''$ and $\mathcal{E}''$) are the subsets of $[0..m]$ (respectively $[m..m+m']$) defined like in the statement of the proposition applied to the evaluation $\boldsymbol{x}_0 = \mathsf{g}(\boldsymbol{x}_{[1..m]})$ (respectively $\boldsymbol{x}_m = \mathsf{h}(\boldsymbol{x}_{[m+1..m+m']})$):

- $\mathcal{A}' = \mathcal{P}(\boldsymbol{x}_{[1..m]})\bigcup\mathcal{I}(\boldsymbol{x}_{\{0\}})$;
- $\mathcal{E}' = \mathcal{I}(\boldsymbol{x}_{[1..m]})\bigcup\mathcal{P}(\boldsymbol{x}_{\{0\}})$;
- $\mathcal{A}'' = \mathcal{P}(\boldsymbol{x}_{[m+1..m+m']})\bigcup\mathcal{I}(\boldsymbol{x}_{\{m\}})$;
- $\mathcal{E}'' = \mathcal{I}(\boldsymbol{x}_{[m+1..m+m']})\bigcup\mathcal{P}(\boldsymbol{x}_{\{m\}})$.

Two cases have to be studied: either $\boldsymbol{x}_m \in \mathbb{IR}$ or $\boldsymbol{x}_m \in \overline{\mathbb{IR}}$. On one hand, if $\boldsymbol{x}_m \in \mathbb{IR}$ then $\mathcal{A} = (\mathcal{A}'\backslash\{m\})\bigcup\mathcal{A}''$ and $\mathcal{E} = \mathcal{E}'\bigcup(\mathcal{E}''\backslash\{m\})$. On the other hand, if $\boldsymbol{x}_m \in \overline{\mathbb{IR}}$ then $\mathcal{A} = \mathcal{A}'\bigcup(\mathcal{A}''\backslash\{m\})$ and $\mathcal{E} = (\mathcal{E}'\backslash\{m\})\bigcup\mathcal{E}''$. Therefore, as $\mathcal{A}'$, $\mathcal{E}'$, $\mathcal{A}''$ and $\mathcal{E}''$ are nonempty by induction hypothesis, $\mathcal{A}$ and $\mathcal{E}$ are also nonempty in both cases. It remains to construct the function $s_{\mathcal{E}}$:

1. If $\boldsymbol{x}_m \in \mathbb{IR}$ then $\mathcal{A} = (\mathcal{A}'\backslash\{m\})\bigcup\mathcal{A}''$ and $\mathcal{E} = \mathcal{E}'\bigcup(\mathcal{E}''\backslash\{m\})$ and $s_{\mathcal{E}}$ is defined by

$$s_{\mathcal{E}'}(x_{\mathcal{A}}) = s'_{\mathcal{E}'}(y_{\mathcal{A}'})$$

where $y_{\mathcal{A}'}$ is defined by $y_{\mathcal{A}'\backslash\{m\}} = x_{\mathcal{A}'\backslash\{m\}}$ and $y_m = s''_m(x_{\mathcal{A}''})$, and

$$s_{\mathcal{E}''\backslash\{m\}}(x_{\mathcal{A}}) = s''_{\mathcal{E}''\backslash\{m\}}(x_{\mathcal{A}''}).$$

2. If $\boldsymbol{x}_m \in \mathbb{IR}$ then $\mathcal{A} = \mathcal{A}'\bigcup(\mathcal{A}''\backslash\{m\})$ and $E = (\mathcal{E}'\backslash\{m\})\bigcup\mathcal{E}''$ and $s_{\mathcal{E}}$ is defined by

$$s_{\mathcal{E}'\backslash\{m\}}(x_{\mathcal{A}}) = s'_{\mathcal{E}'\backslash\{m\}}(x_{\mathcal{A}'})$$

and

$$s_{\mathcal{E}''}(x_{\mathcal{A}}) = s''_{\mathcal{E}''}(y_{\mathcal{A}''})$$

where $y_{\mathcal{A}''}$ is defined by $y_m = s'_m(x_{\mathcal{A}'})$ and $y_{\mathcal{A}''\backslash\{m\}} = x_{\mathcal{A}''\backslash\{m\}}$.

In both cases, the following implication holds:

$$x_{\mathcal{E}} = s_{\mathcal{E}}(x_{\mathcal{A}}) \Longrightarrow \big(\exists x_m \in \text{pro } \boldsymbol{x}_m\big)\big(x_{\mathcal{E}'} = s'_{\mathcal{E}'}(x_{\mathcal{A}'}) \wedge x_{\mathcal{E}''} = s''_{\mathcal{E}''}(x_{\mathcal{A}''})\big).$$

In the first case, $x_m = s''_m(x_{\mathcal{A}''})$ and in the second case, $x_m = s'_m(x_{\mathcal{A}'})$. Therefore, thanks to the definitions of s' and s'', $x_{\mathcal{E}} = s_{\mathcal{E}}(x_{\mathcal{A}})$ implies

$$\big(\exists x_m \in \text{pro } \boldsymbol{x}_m\big)\big(x_0 = g(x_{[1..m]}) \wedge x_m = h(x_{[m+1,m+m']})\big).$$

This eventually implies $x_{\mathcal{E}} = s_{\mathcal{E}}(x_{\mathcal{A}}) \Longrightarrow x_0 = f(x_{\boldsymbol{N}})$. Therefore, the function s satisfies (15). $\qquad\square$

The next corollary will be useful for the proof of Theorem 11.1. It generalizes Proposition 11.1 to vector-valued functions.

Corollary 11.1. *Let $f_{[1..m]}(x_{[1..n]}) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous function and $\mathsf{f}_{[1..m]}$ an expression of this function involving elementary functions of Ω and Φ where **each variable has only one occurrence**[3]. Consider $\boldsymbol{x}_{[1..n]} \in \mathbb{KR}^n$ and evaluate $\boldsymbol{x}_{0k} = \mathsf{f}_k(\boldsymbol{x}_{[1..n]})$. Define $\boldsymbol{M}_0 = \{(0,1),\ldots,(0,m)\}$ and $\mathcal{A} = \mathcal{P}(\boldsymbol{x}_{[1..n]})\bigcup\mathcal{I}(\boldsymbol{M}_0)$ and $\mathcal{E} = \mathcal{I}(\boldsymbol{x}_{[1..n]})\bigcup\mathcal{P}(\boldsymbol{M}_0)$. Then the quantified proposition*

$$\big(\forall x_{\mathcal{A}} \in \text{pro } \boldsymbol{x}_{\mathcal{A}}\big)\big(\exists x_{\mathcal{E}} \in \text{pro } \boldsymbol{x}_{\mathcal{E}}\big)\big(f_{[1..m]}(x_{[1..n]}) = x_{\boldsymbol{M}_0}\big)$$

has a continuous Skolem function.

Remark 16. The vectorial equality $f_{[1..m]}(x_{[1..n]}) = x_{\boldsymbol{M}_0}$ means $f_k(x_{[1..m]}) = x_{0k}$ for all $k \in [1..m]$.

Proof. The proof is trivial. One just has to apply Proposition 11.1 to each $\boldsymbol{x}_{0k} = \mathsf{f}_k(\boldsymbol{x}_{[1..n]})$. $\qquad\square$

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[3]Each variable should have one occurrence over the whole expression of the vectorial function, thus the function components cannot share variables here.

11.2 Natural AE-Extensions

This subsection is dedicated to the construction of natural AE-extensions of vector-valued functions $f = (f_1, \ldots, f_m)^T$. This construction also stands for real-valued functions. In contrast with the classical interval natural extensions, the expression of the function f has to be modified before applying a generalized interval evaluation. Some examples are now presented in order to introduce Theorem 11.1. The following example displays a situation where multiple occurrences can be handle without any modification of the function expression (these situations actually correspond to the classical interval extensions, i.e. to expressions evaluated with proper arguments).

Example 11.1. Consider $f(x) = x - x$ and $\boldsymbol{x} = [-1, 1]$. The generalized interval evaluation raises $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{x} = [-2, 2]$. Clearly $\boldsymbol{z}$ is $(f, \boldsymbol{x})$-interpretable, i.e. the quantified proposition $(\forall x \in \boldsymbol{x})(\exists z \in \boldsymbol{z})(f(x) = z)$ is true. In order to prove this interpretation, one has to use Proposition 11.1. To this end, he has to consider the auxiliary expression $\tilde{f}(x_1, x_2) = x_1 - x_2$ where each occurrence is considered as an independent variable. We have $\boldsymbol{z} = \tilde{f}(\boldsymbol{x}, \boldsymbol{x})$. As $\tilde{f}$ contains only one occurrence of each variable, one can apply Proposition 11.1 which proves that the following quantified proposition is true:

$$\big(\forall x_1 \in \boldsymbol{x}\big)\big(\forall x_2 \in \boldsymbol{x}\big)\big(\exists z \in \boldsymbol{z}\big)\big(\tilde{f}(x_1, x_2) = z\big).$$

Finally, restricting $(x_1, x_2)^T$ to the diagonal of $(\boldsymbol{x}, \boldsymbol{x})^T$ one validates the quantified proposition $(\forall x \in \boldsymbol{x})(\exists z \in \boldsymbol{z})(\tilde{f}(x, x) = z)$. It remains to notice that $\tilde{f}(x, x) = f(x)$.

An example is now presented which shows that a generalized interval evaluation without any modification of the expression does not raise interpretable generalized intervals in general.

Example 11.2. Consider $f(x) = x - x$ and $\boldsymbol{x} = [1, -1]$. The generalized interval evaluation raises $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{x} = [2, -2]$. If $\boldsymbol{z}$ was $(f, \boldsymbol{x})$-interpretable then the quantified proposition $(\forall z \in \text{pro } \boldsymbol{z})(\exists x \in \text{pro } \boldsymbol{x})(f(x) = z)$ would be true. This quantified proposition is actually false. Let us enlighten the reason why this generalized interval evaluation fails to provide a $(f, \boldsymbol{x})$-interpretable interval. As in Example 11.1 the auxiliary expression $\tilde{f}(x_1, x_2) = x_1 - x_2$ has to be used in order to apply Proposition 11.1. Following the argument presented in Example 11.1, the generalized interval evaluation $\boldsymbol{z} = \tilde{f}(\boldsymbol{x}, \boldsymbol{x})$ allows using Proposition 11.1 to validate the quantified proposition

$$\big(\forall z \in \text{pro } \boldsymbol{z}\big)\big(\exists x_1 \in \text{pro } \boldsymbol{x}\big)\big(\exists x_2 \in \text{pro } \boldsymbol{x}\big)\big(\tilde{f}(x_1, x_2) = z\big).$$

This quantified proposition does not imply $(\forall z \in \text{pro } \boldsymbol{z})(\exists x \in \text{pro } \boldsymbol{x})(\tilde{f}(x, x) = z)$ in general (because the two occurrences of x cannot be linked anymore through a diagonal argument) and this explains why $\boldsymbol{z}$ is not $(f, \boldsymbol{x})$-interpretable.

In view of the previous example, the problem that prevents from using the generalized interval evaluation with expressions that contains multiple occurrence of variable is the impossibility of linking two occurrences that are both existentially quantified. The idea to overcome this difficulty is to change the quantifier associated with one occurrence so that the quantified proposition $(\forall x_1 \in \boldsymbol{x})(q^{(\boldsymbol{z})} z \in \boldsymbol{z})(\exists x_2 \in \boldsymbol{x})(\tilde{f}(x_1, x_2) = z)$ is actually validated by the generalized interval evaluation. Then one will be in position to prove that $\boldsymbol{z}$ is $(f, \boldsymbol{x})$-interpretable. This process is illustrated by the next example.

Example 11.3. In the situation of Example 11.2, the auxiliary expression $\mathsf{g}(\boldsymbol{x}) = $ pro $\boldsymbol{x} - \boldsymbol{x}$ is now considered. The generalized interval evaluation now raises $\boldsymbol{z} = \mathsf{g}(\boldsymbol{x}) = [-1,1] - [1,-1] = [0,0]$. Therefore we have to prove that the quantified proposition

$$\big(\forall z \in \text{pro } \boldsymbol{z}\big)\big(\exists x \in \text{pro } \boldsymbol{x}\big)\big(f(x) = z\big) \tag{16}$$

is true. Notice that $\boldsymbol{z} = \tilde{\mathsf{f}}(\text{pro } \boldsymbol{x}, \boldsymbol{x})$. Therefore, applying Proposition 11.1 to the evaluation $\tilde{\mathsf{f}}(\text{pro } \boldsymbol{x}, \boldsymbol{x}) = \boldsymbol{z}$ one proves that the quantified proposition

$$\big(\forall x_1 \in \text{pro } \boldsymbol{x}\big)\big(\forall z \in \text{pro } \boldsymbol{z}\big)\big(\exists x_2 \in \text{pro } \boldsymbol{x}\big)\big(\tilde{\mathsf{f}}(x_1, x_2) = z\big)$$

has a continuous Skolem function, i.e. on proves the existence of a **continuous** function $s : \text{pro } \boldsymbol{z} \times \text{pro } \boldsymbol{x} \longrightarrow \text{pro } \boldsymbol{x}$ that satisfies $x_2 = s(z, x_1) \Longrightarrow \tilde{\mathsf{f}}(x_1, x_2) = z$. Now fix any value $z \in \text{pro } \boldsymbol{z}$ (in the case of this example, pro $\boldsymbol{z}$ contains only one real) so $s(z, .)$ now has pro $\boldsymbol{x}$ as domain and pro $\boldsymbol{x}$ has co-domain. Therefore one can apply the Brouwer fixed point theorem (see Appendix A) that proves the existence of $x \in \text{pro } \boldsymbol{x}$ such that $s(z, x) = x$. As this holds for any $z \in \text{pro } \boldsymbol{z}$ the following quantified proposition is true:

$$\big(\forall z \in \text{pro } \boldsymbol{z}\big)\big(\exists x \in \text{pro } \boldsymbol{x}\big)\big(s(z, x) = x\big).$$

Finally, $s(z, x) = x$ implies $z = \tilde{\mathsf{f}}(x, x)$ which implies $z = f(x)$. Therefore the quantified proposition (16) is proved to hold.

The next theorem generalizes the previous arguments providing the construction of natural AE-extensions.

Theorem 11.1. *Let $f_{[1..m]} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous function and $\mathsf{f}_{[1..m]}$ an expression of $f_{[1..m]}$ involving elementary functions from Ω and Φ. Define the expression $\mathsf{g}_{[1..m]}$ from $\mathsf{f}_{[1..m]}$ by inserting the operation pro before all but one occurrences of each variable (see Example 11.4 and Example 11.5). Then the interval function $\boldsymbol{x}_{[1..n]} \longmapsto \mathsf{g}_{[1..m]}(\boldsymbol{x}_{[1..n]})$ is an AE-extension of $f_{[1..m]}$ which is called a natural AE-extension of f.*

Proof. Denote $[1..n]$ and $[1..m]$ by $\boldsymbol{N}$ and $\boldsymbol{M}$ respectively. First, we clearly have $\mathsf{g}_{\boldsymbol{M}}(x_{\boldsymbol{N}}) = f_{\boldsymbol{M}}(x_{\boldsymbol{N}})$ for all $x \in \mathbb{R}^n$ because pro $x = x$ and the AE-extension $\boldsymbol{h}$ of any elementary function h satisfies $\boldsymbol{h}(x) = h(x)$. It remains to prove that $\mathsf{g}_{\boldsymbol{M}}(\boldsymbol{x}_{\boldsymbol{N}})$ is $(f_{\boldsymbol{M}}, \boldsymbol{x}_{\boldsymbol{N}})$-interpretable for all $\boldsymbol{x}_{\boldsymbol{N}} \in \mathbb{KR}^n$. Consider any $\boldsymbol{x}_{\boldsymbol{N}} \in \mathbb{KR}^n$ and define for all $k \in \boldsymbol{M}$ the interval $\boldsymbol{x}_{0k} = \mathsf{g}_k(\boldsymbol{x}_{\boldsymbol{N}})$. We have to prove that the following quantified proposition is true:

$$\big(\forall x_{\mathcal{A}} \in \text{pro } \boldsymbol{x}_{\mathcal{A}}\big)\big(\exists x_{\mathcal{E}} \in \text{pro } \boldsymbol{x}_{\mathcal{E}}\big)\big(f_{\boldsymbol{M}}(x_{\boldsymbol{N}}) = x_{\boldsymbol{M}_0}\big) \tag{17}$$

where $\boldsymbol{M}_0 = \{(0, k) | k \in \boldsymbol{M}\}$, $\mathcal{A} = \mathcal{P}(\boldsymbol{x}_{\boldsymbol{N}}) \bigcup \mathcal{I}(\boldsymbol{x}_{\boldsymbol{M}_0})$ and $\mathcal{E} = \mathcal{I}(\boldsymbol{x}_{\boldsymbol{N}}) \bigcup \mathcal{P}(\boldsymbol{x}_{\boldsymbol{M}_0})$.

Remark 17. The sets of indices $\boldsymbol{M}$ and $\boldsymbol{M}_0$ have the same cardinality. Therefore the equality $f_{\boldsymbol{M}}(x_{\boldsymbol{N}}) = x_{\boldsymbol{M}_0}$ is well defined and means $f_i(x_{\boldsymbol{N}}) = x_{0i}$ for all $i \in [1..m]$.

In order to use Corollary 11.1, each occurrence of each variable has to be considered as an independent variable. To this end, define the auxiliary expression $\tilde{\mathsf{f}}_{\boldsymbol{M}}$ as the expression obtained from $\mathsf{f}_{\boldsymbol{M}}$ considering as independent variables each occurrence of each variable. Denote the number of occurrences of the variable x_i by n_i and the jth occurrence of the variable x_i by x_{ij}. Define the set of indices $\mathcal{K} = \{(i, j) \in \mathbb{N}^2 | i \in [1..n], j \in [1..n_i]\}$. Provided that all occurrences of each variable has the

same value in the evaluation of $\tilde{\mathsf{f}}_{\boldsymbol{M}}$ then the latter equals to f. In order to formalize this idea, define $\sigma_{\mathcal{K}} : \mathbb{R}^n \longrightarrow \mathbb{R}^{\sum n_i}$ by $\sigma_{ij}(x_{\boldsymbol{N}}) = x_i$. As a consequence, we have $f_{\boldsymbol{M}}(x_{\boldsymbol{N}}) = \tilde{\mathsf{f}}_{\boldsymbol{M}}\big(\sigma_{\mathcal{K}}(x_{\boldsymbol{N}})\big)$ for all $x_{\boldsymbol{N}} \in \mathbb{R}^n$.

The choice of one occurrence of each variable made in the statement of the proposition (where no operation pro is inserted) can now be formalized by a function $\pi : \boldsymbol{N} \longrightarrow \mathcal{K}$. Define $\boldsymbol{x}_{\mathcal{K}} \in \mathbb{KR}^{\sum n_i}$ by $\boldsymbol{x}_{ij} = \boldsymbol{x}_i$ if $\boldsymbol{x}_i$ is proper or $\pi(i) = (i,j)$ and $\boldsymbol{x}_{ij} = \text{pro } \boldsymbol{x}_i$ otherwise. As a consequence of these definitions we have $\mathsf{g}_{\boldsymbol{M}}(\boldsymbol{x}_{\boldsymbol{N}}) = \tilde{\mathsf{f}}_{\boldsymbol{M}}(\boldsymbol{x}_{\mathcal{K}})$ and therefore $\tilde{\mathsf{f}}_{\boldsymbol{M}}(\boldsymbol{x}_{\mathcal{K}}) = \boldsymbol{x}_{\boldsymbol{M}_0}$.

Now, as $\tilde{\mathsf{f}}_{\boldsymbol{M}}$ is compound elementary functions from Ω and Φ and contains only one occurrence of each variable, we can apply Corollary 11.1 to the evaluation $\tilde{\mathsf{f}}(\boldsymbol{x}_{\mathcal{K}}) = \boldsymbol{x}_{\boldsymbol{M}_0}$ which proves the existence of a continuous function $s_{\mathcal{K}_{\mathcal{E}}}$ that satisfies

$$s_{\mathcal{K}_{\mathcal{E}}} : \text{pro } \boldsymbol{x}_{\mathcal{K}_{\mathcal{A}}} \longrightarrow \text{pro } \boldsymbol{x}_{\mathcal{K}_{\mathcal{E}}} \quad \text{s.t.} \quad x_{\mathcal{K}_{\mathcal{E}}} = s_{\mathcal{K}_{\mathcal{E}}}(x_{\mathcal{K}_{\mathcal{A}}}) \implies x_{\boldsymbol{M}_0} = \tilde{\mathsf{f}}_{\boldsymbol{M}}(x_{\mathcal{K}})$$

where $\mathcal{K}_{\mathcal{A}} = \mathcal{P}(\boldsymbol{x}_{\mathcal{K}}) \bigcup \mathcal{I}(\boldsymbol{x}_{\boldsymbol{M}_0})$ and $\mathcal{K}_{\mathcal{E}} = \mathcal{I}(\boldsymbol{x}_{\mathcal{K}}) \bigcup \mathcal{P}(\boldsymbol{x}_{\boldsymbol{M}_0})$.

Denote $\mathcal{K}_{\mathcal{A}} \bigcup \mathcal{K}_{\mathcal{E}}$ (which is equal to $\mathcal{K} \bigcup \boldsymbol{M}_0$) by $\mathcal{L}$ and define the function $\sigma'_{\mathcal{L}} :$ pro $\boldsymbol{x}_{\mathcal{A} \bigcup \mathcal{E}} \longrightarrow$ pro $\boldsymbol{x}_{\mathcal{L}}$ by $\sigma'_{\boldsymbol{M}_0}(x_{\mathcal{A} \bigcup \mathcal{E}}) = x_{\boldsymbol{M}_0}$ (so $\sigma'_{0k}(x_{\mathcal{A} \bigcup \mathcal{E}}) = x_{0k}$ and the co-domain of $\sigma_{0k}(x_{\mathcal{A} \bigcup \mathcal{E}})$ is indeed pro $\boldsymbol{x}_{0k}$) and $\sigma'_{\mathcal{K}}(x_{\mathcal{A} \bigcup \mathcal{E}}) = \sigma_{\mathcal{K}}(x_{\boldsymbol{N}})$ (so $\sigma'_{ij}(x_{\mathcal{A} \bigcup \mathcal{E}}) = x_i$ by definition of $\sigma_{\mathcal{K}}$ and the co-domain of $\sigma_{ij}(x_{\mathcal{A} \bigcup \mathcal{E}})$ is pro $\boldsymbol{x}_i$ which equals pro $\boldsymbol{x}_{ij}$ by definition of $\boldsymbol{x}_{ij}$). Let us display two properties of $\sigma'_{\mathcal{L}}$: fix any $x_{\mathcal{A} \bigcup \mathcal{E}} \in$ pro $\boldsymbol{x}_{\mathcal{A} \bigcup \mathcal{E}}$ then

Claim one: $\tilde{\mathsf{f}}_{\boldsymbol{M}}(\sigma'_{\mathcal{K}}(x_{\mathcal{A} \bigcup \mathcal{E}})) = f_{\boldsymbol{M}}(x_{\boldsymbol{N}})$. Indeed, $\tilde{\mathsf{f}}_{\boldsymbol{M}}(\sigma'_{\mathcal{K}}(x_{\mathcal{A} \bigcup \mathcal{E}})) = \tilde{\mathsf{f}}_{\boldsymbol{M}}(\sigma_{\mathcal{K}}(x_{\boldsymbol{N}}))$ by definition of $\sigma'_{\mathcal{K}}$ and $f_{\boldsymbol{M}}(x_{\boldsymbol{N}}) = \tilde{\mathsf{f}}_{\boldsymbol{M}}\big(\sigma_{\mathcal{K}}(x_{\boldsymbol{N}})\big)$ as seen previously.

Claim two: $\sigma'_{\mathcal{K}_{\mathcal{E}}}(x_{\mathcal{A} \bigcup \mathcal{E}}) = x_{\mathcal{E}}$. By definition, $\mathcal{E} = \mathcal{I}(\boldsymbol{x}_{\boldsymbol{N}}) \bigcup \mathcal{P}(\boldsymbol{x}_{\boldsymbol{M}_0})$ and $\mathcal{K}_{\mathcal{E}} = \mathcal{I}(\boldsymbol{x}_{\mathcal{K}}) \bigcup \mathcal{P}(\boldsymbol{x}_{\boldsymbol{M}_0})$. Let us denote temporarily $\mathcal{I}(\boldsymbol{x}_{\boldsymbol{N}}) = \{e_1, \ldots, e_s\}$ (where $e_k \in \mathbb{N}$) and $\mathcal{P}(\boldsymbol{x}_{\boldsymbol{M}_0}) = \{m_1, \ldots, m_t\}$ (where $m_k \in \mathbb{N}^2$). By definition of $\boldsymbol{x}_{\mathcal{K}}$, $\boldsymbol{x}_{ij}$ is improper if and only if $i \in \mathcal{I}(\boldsymbol{x}_{\boldsymbol{N}})$ (i.e. x_i is existentially quantified) and $\pi(i) = (i,j)$ (i.e. x_{ij} is the occurrence of x_i which is not preceded by an operation pro in the expression of g). As a consequence, we have $\mathcal{I}(\boldsymbol{x}_{\mathcal{K}}) = \{\pi(e_1), \ldots, \pi(e_s)\}$. Respecting the lexicographic order, we have $\mathcal{E} = \{m_1, \ldots, m_t, e_1, \ldots, e_s\}$ and $\mathcal{K}_{\mathcal{E}} = \{m_1, \ldots, m_t, \pi(e_1), \ldots, \pi(e_s)\}$. Therefore, writing the equality $\sigma'_{\mathcal{K}_{\mathcal{E}}}(x_{\mathcal{A} \bigcup \mathcal{E}}) = x_{\mathcal{E}}$ componentwise, we just have to prove that $\sigma'_{m_k}(x_{\mathcal{A} \bigcup \mathcal{E}}) = x_{m_k}$ for $k \in [1..t]$ and $\sigma'_{\pi(e_k)}(x_{\mathcal{A} \bigcup \mathcal{E}}) = x_{e_k}$ for $k \in [1..s]$. This is actually true because of the definitions of $\sigma'_{\mathcal{L}}$ and $\sigma_{\mathcal{K}}$.

Now define $s'_{\mathcal{E}} :$ pro $\boldsymbol{x}_{\mathcal{A} \bigcup \mathcal{E}} \longrightarrow$ pro $\boldsymbol{x}_{\mathcal{E}}$ by $s'_{\mathcal{E}}(x_{\mathcal{A} \bigcup \mathcal{E}}) = s_{\mathcal{K}_{\mathcal{E}}}\big(\sigma'_{\mathcal{K}_{\mathcal{A}}}(x_{\mathcal{A} \bigcup \mathcal{E}})\big)$. Now as by construction $\mathcal{A} \bigcap \mathcal{E} = \emptyset$, fixing $x_{\mathcal{A}}$ to any value in pro $\boldsymbol{x}_{\mathcal{A}}$ the function $s'_{\mathcal{E}}$ then has pro $\boldsymbol{x}_{\mathcal{E}}$ as domain and pro $\boldsymbol{x}_{\mathcal{E}}$ as co-domain. Therefore we can apply the Brouwer fixed point theorem (see Appendix A) that proves the existence of $x_{\mathcal{E}} \in$ pro $\boldsymbol{x}_{\mathcal{E}}$ such that $s'_{\mathcal{E}}(x_{\mathcal{A} \bigcup \mathcal{E}}) = x_{\mathcal{E}}$. As this holds for any $x_{\mathcal{A}} \in$ pro $\boldsymbol{x}_{\mathcal{A}}$, the following quantified proposition is true:

$$\big(\forall x_{\mathcal{A}} \in \text{pro } \boldsymbol{x}_{\mathcal{A}}\big)\big(\exists x_{\mathcal{E}} \in \text{pro } \boldsymbol{x}_{\mathcal{E}}\big)\big(s'_{\mathcal{E}}(x_{\mathcal{A} \bigcup \mathcal{E}}) = x_{\mathcal{E}}\big). \tag{18}$$

We now finally prove that $s'_{\mathcal{E}}(x_{\mathcal{A} \bigcup \mathcal{E}}) = x_{\mathcal{E}}$ implies $x_{\boldsymbol{M}_0} = f_{\boldsymbol{M}}(x_{\boldsymbol{N}})$. Suppose that $s'_{\mathcal{E}}(x_{\mathcal{A} \bigcup \mathcal{E}}) = x_{\mathcal{E}}$. Then by definition of $s'_{\mathcal{E}}$ we have $s_{\mathcal{K}_{\mathcal{E}}}(\sigma'_{\mathcal{K}_{\mathcal{A}}}(x_{\mathcal{A} \bigcup \mathcal{E}})) = x_{\mathcal{E}}$. As by *Claim two* $x_{\mathcal{E}} = \sigma'_{\mathcal{K}_{\mathcal{E}}}(x_{\mathcal{A} \bigcup \mathcal{E}})$, we have $s_{\mathcal{K}_{\mathcal{E}}}(\sigma'_{\mathcal{K}_{\mathcal{A}}}(x_{\mathcal{A} \bigcup \mathcal{E}})) = \sigma'_{\mathcal{K}_{\mathcal{E}}}(x_{\mathcal{A} \bigcup \mathcal{E}})$. Using the property satisfied by $s_{\mathcal{K}_{\mathcal{E}}}$ we obtain $\sigma'_{\boldsymbol{M}_0}(x_{\mathcal{A} \bigcup \mathcal{E}}) = \tilde{\mathsf{f}}_{\boldsymbol{M}}(\sigma'_{\mathcal{K}}(x_{\mathcal{A} \bigcup \mathcal{E}}))$. As $\sigma'_{\boldsymbol{M}_0}(x_{\mathcal{A} \bigcup \mathcal{E}}) = x_{\boldsymbol{M}_0}$ by definition of $\sigma'_{\mathcal{L}}$, we have $x_{\boldsymbol{M}_0} = \tilde{\mathsf{f}}_{\boldsymbol{M}}(\sigma'_{\mathcal{K}}(x_{\mathcal{A} \bigcup \mathcal{E}}))$. That is

$x_{\boldsymbol{M}_0} = f_{\boldsymbol{M}}(x_{\boldsymbol{N}})$ thanks to *Claim one*. Therefore, the quantified proposition (18) implies the quantified proposition (17). $\square$

A real functions has several natural AE-extensions, depending on the way the operations pro are inserted. This is illustrated by the following two examples.

Example 11.4. Consider the function $f(x, y) = xy + x(x + y)$. The following interval functions are natural AE-extensions of f:

$$\begin{array}{rcl}
\boldsymbol{xy} + (\text{pro } \boldsymbol{x})(\text{pro } \boldsymbol{x} + \text{pro } \boldsymbol{y}) & ; & \boldsymbol{x}(\text{pro } \boldsymbol{y}) + (\text{pro } \boldsymbol{x})(\text{pro } \boldsymbol{x} + \boldsymbol{y}) \\
(\text{pro } \boldsymbol{x})\boldsymbol{y} + \boldsymbol{x}(\text{pro } \boldsymbol{x} + \text{pro } \boldsymbol{y}) & ; & (\text{pro } \boldsymbol{x})(\text{pro } \boldsymbol{y}) + \boldsymbol{x}(\text{pro } \boldsymbol{x} + \boldsymbol{y}) \\
(\text{pro } \boldsymbol{x})\boldsymbol{y} + (\text{pro } \boldsymbol{x})(\boldsymbol{x} + \text{pro } \boldsymbol{y}) & ; & (\text{pro } \boldsymbol{x})(\text{pro } \boldsymbol{y}) + (\text{pro } \boldsymbol{x})(\boldsymbol{x} + \boldsymbol{y})
\end{array}$$

Example 11.5. Consider the function $f(x, y) = (xy \,,\, x(x + y))^T$. Its natural AE-extensions are

$$\begin{array}{rcl}
\big(\boldsymbol{xy} \,,\, (\text{pro } \boldsymbol{x})(\text{pro } \boldsymbol{x} + \text{pro } \boldsymbol{y})\big)^T & ; & \big(\boldsymbol{x}(\text{pro } \boldsymbol{y}) \,,\, (\text{pro } \boldsymbol{x})(\text{pro } \boldsymbol{x} + \boldsymbol{y})\big)^T \\
\big((\text{pro } \boldsymbol{x})\boldsymbol{y} \,,\, \boldsymbol{x}(\text{pro } \boldsymbol{x} + \text{pro } \boldsymbol{y})\big)^T & ; & \big((\text{pro } \boldsymbol{x})(\text{pro } \boldsymbol{y}) \,,\, \boldsymbol{x}(\text{pro } \boldsymbol{x} + \boldsymbol{y})\big)^T \\
\big((\text{pro } \boldsymbol{x})\boldsymbol{y} \,,\, (\text{pro } \boldsymbol{x})(\boldsymbol{x} + \text{pro } \boldsymbol{y})\big)^T & ; & \big((\text{pro } \boldsymbol{x})(\text{pro } \boldsymbol{y}) \,,\, (\text{pro } \boldsymbol{x})(\boldsymbol{x} + \boldsymbol{y})\big)^T
\end{array}$$

If the expression of f contains only one occurrence of each variable then no operation pro is inserted. E.g. the natural AE-extension of $f(x_{[1..n]}, y_{[1..n]}) = \sum x_k y_k$ is $f(\boldsymbol{x}_{[1..n]}, \boldsymbol{y}_{[1..n]}) = \sum \boldsymbol{x}_k \boldsymbol{y}_k$. Finally, the set of elementary functions Ω and Φ cannot be extended without taking care that the new functions satisfy Proposition 10.1 and Proposition 10.2. This is illustrated by the counter example presented in Appendix B (page 177).

12 On the Quality of the Natural AE-Extension

Two theoretical results on the quality of natural AE-extensions are now provided.

12.1 Order of Convergence

First, the order of convergence of natural AE-extensions is investigated. The statement of Theorem 12.1 is an extension of a well known results in classical interval theory.

Lemma 12.1. *The operation* pro $: \mathbb{KR} \longrightarrow \mathbb{KR}$ *is Lipschitz continuous, and therefore locally Lipschitz continuous.*

Proof. It is clear that $\text{dist}(\text{pro } \boldsymbol{x}, \text{pro } \boldsymbol{y}) \leq \text{dist}(\boldsymbol{x}, \boldsymbol{y})$, that is, pro is Lipschitz continuous. $\square$

Theorem 12.1. *The natural AE-extensions have a linear convergence excepted if its expression contains some $\sqrt[n]{x}$ which has to be evaluated at 0 (see next remark).*

Proof. In this case, the natural AE-extension is composed on one hand of elementary AE-extensions which are locally Lipschitz continuous (see [14]) and on the other hand of the operation pro which is also locally Lipschitz continuous. So the natural AE-extension is locally Lipschitz continuous. This eventually implies a linear convergence thanks to the Proposition 7.2. $\square$

Remark 18. As pointed out in [25], the elementary functions $\sqrt[n]{x}$ are not locally Lipschitz continuous inside $[0, +\infty[$ because they have an infinite derivative at 0 (however, they are locally Lipschitz continuous inside $]0, +\infty[$). Therefore, a natural AE-extension which involves such an elementary function which has to be evaluated at 0 may not have a linear convergence. Check for example that the natural AE-extension of $f(x) = \sqrt{\text{abs } x}$ does not have a linear convergence inside $\boldsymbol{x}^{\text{ref}} = [-1, 1]$.

12.2 Minimality

Contrary to the classical interval natural extension, an expression which contains only one occurrence of each variable does not compute a minimal $(f, \boldsymbol{x})$-interpretable interval in general. The next example is taken from [34].

Example 12.1. Consider $f(x) = (x_1 + x_2)(x_3 + x_4)$ and the generalized interval $\boldsymbol{x} = ([-2, 2], [1, -1], [-1, 1], [2, -2])^T$. In this case, $f^*(\boldsymbol{x}) = [1.5, -1.5]$ so the quantified proposition

$$\big(\forall x_1 \in \boldsymbol{x}_1\big)\big(\forall x_3 \in \boldsymbol{x}_3\big)\big(\forall z \in [-1.5, 1.5]\big)$$
$$\big(\exists x_2 \in \text{pro } \boldsymbol{x}_2\big)\big(\exists x_4 \in \text{pro } \boldsymbol{x}_4\big)\big(f(x) = z\big)$$

is true. The natural AE-extension leads to $(\boldsymbol{x}_1 + \boldsymbol{x}_2)(\boldsymbol{x}_3 + \boldsymbol{x}_4) = [0, 0]$ which is less accurate and allows to validate the less informative quantified proposition

$$\big(\forall x_1 \in \boldsymbol{x}_1\big)\big(\forall x_3 \in \boldsymbol{x}_3\big)\big(\exists x_2 \in \text{pro } \boldsymbol{x}_2\big)\big(\exists x_4 \in \text{pro } \boldsymbol{x}_4\big)\big(f(x) = 0\big). \qquad (19)$$

No explanation for this kind of situations was proposed in [34]. An explanation is now proposed thanks to the proof of Proposition 11.1: Proposition 11.1 applied to the computation $(\boldsymbol{x}_1 + \boldsymbol{x}_2)(\boldsymbol{x}_3 + \boldsymbol{x}_4) = [0, 0]$ proves that the quantified proposition (19) has a continuous Skolem function, i.e. proves that there exists a continuous function

$$s_{\{2,4\}} : (\boldsymbol{x}_1, \boldsymbol{x}_3)^T \longrightarrow (\boldsymbol{x}_2, \boldsymbol{x}_4)^T$$

which satisfies

$$(x_2, x_4)^T = s_{\{2,4\}}(x_1, x_3) \Longrightarrow f(x) = 0.$$

However, when one looks inside the proof of Proposition 11.1, he notices that the function s_2 actually does not depend on x_3 (see the introducing example of Subsection 11.1 page 165 where the expression of $s_{\{2,4\}}$ is provided). Therefore, the choice of x_2 in the quantified proposition (19) actually does not depend on the values of x_3. Therefore, the quantified proposition which is validated by the computation $(\boldsymbol{x}_1 + \boldsymbol{x}_2)(\boldsymbol{x}_3 + \boldsymbol{x}_4) = [0, 0]$ is actually

$$\big(\forall x_1 \in \boldsymbol{x}_1\big)\big(\exists x_2 \in \text{pro } \boldsymbol{x}_2\big)\big(\forall x_3 \in \boldsymbol{x}_3\big)\big(\exists x_4 \in \text{pro } \boldsymbol{x}_4\big)\big(f(x) = 0\big).$$

This explains why this natural AE-extension is not minimal although the involved expression has one occurrence of each variable: it actually solves a more difficult problem!

Remark 19. This more accurate interpretation of the generalized interval evaluation may be useful in practice but this has not yet been studied.

Finally, next proposition states that the natural AE-extension of bilinear real functions is minimal.

Theorem 12.2. *Consider the bilinear function*

$$f : \mathbb{R}^{2n} \longrightarrow \mathbb{R} \quad ; \quad (x, y) \longmapsto \sum_{i \in [1..n]} x_i y_i$$

The natural AE-extension of f is minimal.

Proof. Define $\boldsymbol{N} = [1..2n]$ so

$$f(x_{\boldsymbol{N}}) = \sum_{i \in [1..n]} x_i x_{i+n}.$$

Consider any $\boldsymbol{x_N} \in \mathbb{KR}^{2n}$. The following equality has to be established:

$$f^*(\boldsymbol{x_N}) = \sum_{i \in [1..n]} \boldsymbol{x}_i \boldsymbol{x}_{i+n}$$

Suppose that $\boldsymbol{x}_i$ is proper for $i \in [1..n]$ and $\boldsymbol{x}_i$ is improper for $i \in [n+1, 2n]$ so that $\boldsymbol{x}_i \boldsymbol{x}_{i+n}$ is the product of a proper and an improper interval. The other cases (involving products of proper intervals and products of improper intervals) are similar and simpler. By definition,

$$f^*(\boldsymbol{x_N}) = \Big[\min_{x_{\mathcal{P}} \in \boldsymbol{x}_{\mathcal{P}}} \max_{x_{\mathcal{I}} \in \boldsymbol{x}_{\mathcal{I}}} \sum_{i \in [1..n]} x_i x_{i+n} \; , \; \max_{x_{\mathcal{P}} \in \boldsymbol{x}_{\mathcal{P}}} \min_{x_{\mathcal{I}} \in \boldsymbol{x}_{\mathcal{I}}} \sum_{i \in [1..n]} x_i x_{i+n} \Big] \tag{20}$$

where $\mathcal{P} = \mathcal{P}(\boldsymbol{x_N}) = [1..n]$ and $\mathcal{I} = \mathcal{I}(\boldsymbol{x_N}) = [n+1..2n]$. Also, applying the definition of the AE-multiplication and the formula of the AE-addition, $\sum_{i \in [1..n]} \boldsymbol{x}_i \boldsymbol{x}_{i+n}$ is equal to

$$\Big[\sum_{i \in [1..n]} \min_{x_i \in \boldsymbol{x}_i} \max_{x_{i+n} \in \boldsymbol{x}_{i+n}} x_i x_{i+n} \; , \; \sum_{i \in [1..n]} \max_{x_i \in \boldsymbol{x}_i} \min_{x_{i+n} \in \boldsymbol{x}_{i+n}} x_i x_{i+n} \Big]. \tag{21}$$

The generalized intervals (20) and (21) must be proved to be equal. This equality is proved without any difficulty noticing the following two properties

$$\max_{u \in \boldsymbol{u}} \big(a(u) + b \big) = \big(\max_{u \in \boldsymbol{u}} a(u) \big) + b$$

and

$$\min_{u \in \boldsymbol{u}} \big(a(u) + b \big) = \big(\min_{u \in \boldsymbol{u}} a(u) \big) + b$$

where $a : \mathbb{R} \longrightarrow \mathbb{R}$ and $b \in \mathbb{R}$. $\qquad\square$

Some additional functions may be proved to have a minimal natural AE-extension using a similar proof. The case of bilinear functions is displayed because their minimality has interesting consequence. E.g. the central necessary and sufficient condition provided by Theorem 5.1 of [33] for the membership to linear AE-solution sets can be obtained thanks to the Theorem 12.2.

13 On the Scope of AE-Extensions

The natural AE-extensions are a reformulation of the modal "theorems of $*$ and $**$ interpretation of a modal rational extension" and "theorems of coercion to $*$ and $**$ interpretability" and their n-dimensional versions. Therefore the applications which were proposed in the context of modal intervals are in the scope of AE-extensions. The next examples focuses on some positive and negatives aspects of the natural AE-extensions.

13.1 Application to $n \times n$ Systems of Equations

First, an example is presented where the natural AE-extension succeeds in proving the existence of a solution to a 2×2-system of equations.

Example 13.1. Consider the function

$$f(x, y) = \begin{pmatrix} 81x^2 + y^2 + 18xy - 100 \\ x^2 + 81y^2 + 18xy - 100 \end{pmatrix}$$

One of its natural AE-extension is

$$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) = \begin{pmatrix} 81\boldsymbol{x}^2 & + & (\text{pro } \boldsymbol{y})^2 & + & 18(\text{pro } \boldsymbol{x})(\text{pro } \boldsymbol{y}) - 100 \\ (\text{pro } \boldsymbol{x})^2 & + & 81\boldsymbol{y}^2 & + & 18(\text{pro } \boldsymbol{x})(\text{pro } \boldsymbol{y}) - 100 \end{pmatrix}$$

If $\boldsymbol{x} = \boldsymbol{y} = [1.1, 0.9]$ this natural AE-extension leads to

$$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) = (\ [13.4, -11, 4]\ ,\ [13.4, -11, 4]\ )^T$$

and therefore proves that pro $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) \subseteq \text{range}\,(f, \text{pro } \boldsymbol{x}, \text{pro } \boldsymbol{y})$. In particular, the system of equations $f(x, y) = 0$ is proved to have a solution inside $(\boldsymbol{x}, \boldsymbol{y})^T$.

Classical methods (like the Miranda theorem or interval Newton operators, see e.g. [25]) also succeed in proving the existence of a solution for the previous example. As it was explained in the previous section, the natural AE-extension splits the problem

$$\big(\exists x \in \boldsymbol{x}\big)\big(\exists y \in \boldsymbol{y}\big)\big(f_1(x, y) = 0 \wedge f_2(x, y) = 0\big)$$

into two sub-problems, for example

$$\big(\forall y \in \boldsymbol{y}\big)\big(\exists x \in \boldsymbol{x}\big)\big(f_1(x, y) = 0\big) \quad \text{and} \quad \big(\forall x \in \boldsymbol{x}\big)\big(\exists y \in \boldsymbol{y}\big)\big(f_2(x, y) = 0\big)$$

However, this decomposition is not efficient in general. The next example provides a situation where the natural AE-extension cannot prove the existence of a solution.

Example 13.2. Consider the linear function

$$f(x, y) = \begin{pmatrix} 2y + x \\ 2y - x \end{pmatrix}$$

and the intervals $\boldsymbol{x} = [-\epsilon, \epsilon]$ and $\boldsymbol{y} = [-\epsilon, \epsilon]$ for $\epsilon > 0$. Both f_1 and f_2 satisfy

$$\big(\forall x \in \boldsymbol{x}\big)\big(\exists y \in \boldsymbol{y}\big)\big(f_k(x, y) = 0\big)$$

and none satisfies

$$\big(\forall y \in \boldsymbol{y}\big)\big(\exists x \in \boldsymbol{x}\big)\big(f_k(x, y) = 0\big)$$

So whatever is $\epsilon > 0$, the natural AE-extension will never be able to prove the existence of a solution to the system $f(x, y) = 0$ although $(0, 0)^T$ is a solution to this simple system.

The classical approach to such a problem is to precondition the equation so as to obtain a near-identity equivalent system: the preconditioned system $g(x, y) = Mf(x, y)$ is considered, where M is a non-singular real matrix. So $f(x, y) = 0 \iff g(x, y) = 0$ and M can be chosen so that g is close to the identity map. However, such a preconditioning process drastically increases the number of occurrences, as $g_k(x, y) = M_{k1}f_1(x, y) + M_{k2}f_2(x, y)$. In the classical interval theory, a preconditioning process is usually coupled with a linearization process (e.g. a mean-value extension) which efficiently decreases the influence of the growing number of occurrences.

13.2   Application to Parametric Under-Constrained Systems of Equations

A situation where the AE-extensions have a promising potential is the study of parametric under-constrained systems of equations. The next example illustrates this idea.

Example 13.3. Consider the function $f(a, x) = a_1 x_1 + a_2 x_2 + a_3 x_3 - 5$ and the intervals

$$\boldsymbol{x} = ([0, 2], [0, 2], [0, 2])^T \quad \text{and} \quad \boldsymbol{a} = ([0.9, 1.1], [0.9, 1.1], [0.9, 1.1])^T.$$

We want to prove that

$$\big(\forall a \in \boldsymbol{a}\big)\big(\exists x \in \boldsymbol{x}\big)\big(f(a, x) = 0\big).$$

We can use the natural AE-extension of f evaluated at $(\boldsymbol{a}, \text{dual } \boldsymbol{x})^T$:

$$\begin{aligned}
\boldsymbol{f}(\boldsymbol{a}, \text{dual } \boldsymbol{x}) &= \boldsymbol{a}_1(\text{dual } \boldsymbol{x}_1) + \boldsymbol{a}_2(\text{dual } \boldsymbol{x}_2) + \boldsymbol{a}_3(\text{dual } \boldsymbol{x}_3) \\
&= [0.4, -5]
\end{aligned}$$

This proves the following quantified proposition:

$$\big(\forall a \in \boldsymbol{a}\big)\big(\forall z \in [-5, 0.4]\big)\big(\exists x \in \boldsymbol{x}\big)\big(f(a, x) = z\big)$$

which indeed implies the first quantified proposition.

The classical techniques based on the Miranda theorem or the interval Newton operators do not easily handle parametric under-constrained systems. This academic example illustrates that AE-extensions can provide promising tools in such situations.

13.3   Contractors for Quantified Constraints

An other promising application of AE-extensions is proposed in [10] (in the framework of the original formulation of modal interval theory). It consists of contracting a domain without loosing any solution to a quantified constraint. Consider for example a quantified constraint ϕ on $x \in \mathbb{R}$ like

$$\phi(x) \iff \big(\forall u \in \boldsymbol{u}\big)\big(\exists v \in \boldsymbol{v}\big)\big(f(u, v, x) \geq 0\big).$$

The aim of a contractor is to prove that a given interval $\boldsymbol{x}$ does not contain any real that satisfies $\phi(x)$, i.e. to prove that

$$\big(\forall x \in \boldsymbol{x}\big)\big(\exists u \in \boldsymbol{u}\big)\big(\forall v \in \boldsymbol{v}\big)\big(f(u, v, x) < 0\big). \tag{22}$$

It is proved in [10] that a sufficient condition of the quantified proposition (22) is

$$\neg\Big(\big(\forall u \in \boldsymbol{u}\big)\big(\exists x \in \boldsymbol{x}\big)\big(\exists v \in \boldsymbol{v}\big)\big(f(u, v, x) \geq 0\big)\Big), \tag{23}$$

Finally, a sufficient condition for (23) can be obtained using the optimal AE-extension f^* of f: it is proved in [10] that

$$f^*(\boldsymbol{u}, \text{dual } \boldsymbol{v}, \text{dual } \boldsymbol{x}) \not\subseteq [0, +\infty[^4$$

---

[4]Infinite generalized intervals are not formally defined, but extending the inclusion to such intervals does not present any difficulty.

implies (23). A a consequence, one can discard a box $\boldsymbol{x}$ provided that he proves that the previous inclusion is false. This can be done computing an inner approximation of $f^*(\boldsymbol{u}, \mathrm{dual}\ \boldsymbol{v}, \mathrm{dual}\ \boldsymbol{x})$, i.e. an interval $\boldsymbol{z}$ that satisfies $\boldsymbol{z} \subseteq f^*(\boldsymbol{u}, \mathrm{dual}\ \boldsymbol{v}, \mathrm{dual}\ \boldsymbol{x})$. Such an interval cannot be computed using the developments exposed in the presented paper (as the present paper is dedicated to the computation of intervals $\boldsymbol{z}$ that satisfies $f^*(\boldsymbol{u}, \mathrm{dual}\ \boldsymbol{v}, \mathrm{dual}\ \boldsymbol{x}) \subseteq \boldsymbol{z}$). In [10] a branch and prune algorithm is proposed in order to compute such intervals $\boldsymbol{z}$. In "Modal Intervals Revisited Part 2: A Generalized Interval Mean-Value Extension", a new way to compute such intervals will be proposed, relying on some linearization of f.

14   Conclusion

Modal interval theory has been reformulated in the context of generalized intervals (intervals whose bounds are not constrained to be ordered). New extensions to generalized intervals have been defined, called AE-extensions. These AE-extensions provide the same interpretations than the extensions to modal intervals. The following differences with modal interval theory can be pointed out:

The construction of AE-extensions is similar to the construction of extensions to classical intervals. In particular, the central modal "theorems of $*$ and $**$ interpretation of a modal rational extension" and "theorems of coercion to $*$ and $**$ interpretability" and their n-dimensional versions are summarized in natural AE-extensions. Furthermore, all the defined concepts are generalizations of their classical counterparts: the definition of the minimality of AE-extensions is more general than the optimality of modal interval extensions. On one hand the new definition for minimality is natural and generalizes the definition of minimality of classical interval extensions. On the other hand, it allows to introduce the concept of order of convergence, which is widely used in classical interval theory in order to measure the quality of an extension. Like in the context of extensions to classical intervals, natural AE-extensions have been proved to have a linear convergence (provided that no function $\sqrt[n]{x}$ is evaluated at zero).

The proofs proposed in this paper are new. They intend to fill some gaps which were found in the proof of the modal "theorems of $*$ and $**$ interpretation of a modal rational extension" and their n-dimensional versions.

Finally, the modal "theorems of $*$ and $**$ partially optimal coercion" have not been cast into the new formulation of modal interval theory. They would allow to introduce new AE-extensions which would be more accurate than natural AE-extensions thanks to the study of the monotonicity of the functions. This choice is motivated by the possibility to introduce a mean-value AE-extension in the new formulation of modal interval theory. This is the subject of a the second part of the paper.

Acknowledgments

A   The Brouwer Fixed Point Theorem

The Brouwer fixed point theorem is a famous classical existence theorem (see for example [11] or [25]).

Theorem (Brouwer fixed point theorem)**.** *Let $E \subseteq \mathbb{R}^n$ be nonempty, compact and convex, and $f : E \longrightarrow E$ be continuous. Then, there exists $x \in E$ such that $f(x) = x$.*

B   A Counterexample

Consider the operation $\odot$ defined by $x \odot y := 1 - (x - y)^2$ as an elementary function of Ω. It is now proved that this conducts to a wrong assertion. The generalized interval operation $\boldsymbol{x} \odot \boldsymbol{y}$ is defined as the optimal AE-extension of $x \odot y$. Now consider the continuous function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$f(x, y) = \begin{pmatrix} x \odot y \\ x - y \end{pmatrix}$$

and the intervals $\boldsymbol{x} = [1, -1]$ and $\boldsymbol{y} = [1, -1]$. One of its natural AE-extension is

$$\boldsymbol{z} = \begin{pmatrix} (\text{pro } \boldsymbol{x}) \odot \boldsymbol{y} \\ \boldsymbol{x} - (\text{pro } \boldsymbol{y}) \end{pmatrix}.$$

The interval $(\text{pro } \boldsymbol{x}) \odot \boldsymbol{y}$ is computed in the following way:

$$
\begin{aligned}
(\text{pro } \boldsymbol{x}) \odot \boldsymbol{y} &= \quad \vee_{x \in \text{pro } \boldsymbol{x}} \wedge_{y \in \boldsymbol{y}} x \odot y \\
&= \quad \underset{x \in \text{pro } \boldsymbol{x}}{\vee} \text{dual range} (f_1, x, \text{pro } \boldsymbol{y}) \\
&= \quad \underset{x \in \text{pro } \boldsymbol{x}}{\vee} [1 \ , \ 1 - (1 + |x|)^2] \\
&= \quad [1 \ , \ \underset{x \in \text{pro } \boldsymbol{x}}{\max} 1 - (1 + |x|)^2] \\
&= \quad [1, 0].
\end{aligned}
$$

So, the natural AE-extension evaluation leads to $\boldsymbol{z} = ([1, 0], [0, 0])^T$ and the following quantified proposition must be true:

$$\big(\forall z_1 \in [0, 1]\big)\big(\forall z_2 \in [0, 0]\big)\big(\exists y \in \text{pro } \boldsymbol{y}\big)\big(\exists x \in \text{pro } \boldsymbol{x}\big)\big(f(x, y) = z\big).$$

However, this latter quantified proposition is false: indeed it implies

$$\big(\exists y \in \text{pro } \boldsymbol{y}\big)\big(\exists x \in \text{pro } \boldsymbol{x}\big)\big(f(x, y) = 0\big).$$

However $f_2(x, y) = 0$ implies $x = y$, and therefore $f_1(x, y) = 0$ implies $1 - (x - x)^2 = 0$, which is absurd.

Remark 20. The papers describing modal interval theory do not focus on such cases. For example, some care should be given when applying the results of the section III.4 of [34] so as to be sure that the situation displayed in this section cannot occur.

The wrong conclusion obtained considering $\odot$ as an elementary operation is explained in the following way: the computations $(\text{pro } \boldsymbol{x}) \odot \boldsymbol{y} = [1, 0]$ and $\boldsymbol{x} - (\text{pro } \boldsymbol{y}) = [0, 0]$ validates both following quantified propositions:

$$\big(\forall x \in \text{pro } \boldsymbol{x}\big)\big(\exists y \in \text{pro } \boldsymbol{y}\big)\big(x \odot y = 0\big)$$

$$\text{and} \quad \big(\forall y \in \text{pro } \boldsymbol{y}\big)\big(\exists x \in \text{pro } \boldsymbol{x}\big)\big(x - y = 0\big).$$

However, these two quantified propositions do not imply

$$\big(\exists x \in \text{pro } \boldsymbol{x}\big)\big(\exists y \in \text{pro } \boldsymbol{y}\big)\big(x \odot y = 0 \wedge x - y = 0\big).$$

This is due to the fact that in the first quantified proposition, the choice of the values of y cannot be done continuously with respect to x (and that is the reason why $\odot$ cannot be considered as a elementary function of Ω).

C   Proofs of Some Propositions from Section 10

First of all, the following case is trivial for all functions (one variable or two variables elementary functions):

Proposition C.1. *Let $f : \boldsymbol{x}_{[1..n]} \longrightarrow \mathbb{R}$ be a continuous function where $\boldsymbol{x}_{[1..n]} \in \mathbb{IR}^n$. Define $\boldsymbol{x}_0 = f^*(\boldsymbol{x}_{[1..n]})$ which is proper. Then there exists a* **continuous** *function $s : \boldsymbol{x}_{[1..n]} \longrightarrow \boldsymbol{x}_0$ which satisfies*

$$x_0 = s(x_{[1..n]}) \Longrightarrow x_0 = f(x_{[1..n]})$$

Proof. The function s is nothing but f and is therefore continuous. $\square$

One variable elementary functions

In the case of one variable functions, there are only two cases: either $\boldsymbol{x} \in \mathbb{IR}$ or $\boldsymbol{x} \in \overline{\mathbb{IR}}$. The first case has already been treated by the Proposition C.1, so it remains to study the second one. The next proposition stands for the functions $\exp x$, $\ln x$, $\tan$, $\arccos x$, $\arcsin x$, $\arctan x$, x^n for n odd and, $x^{\frac{1}{n}}$.

Proposition C.2. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous and strictly increasing or decreasing and $\boldsymbol{x} \in \overline{\mathbb{IR}}$. Define $\boldsymbol{z} = f^*(\boldsymbol{x})$, which is improper. Then there exists a* **continuous** *function $s : \text{pro } \boldsymbol{z} \longrightarrow \text{pro } \boldsymbol{x}$ which satisfies $x = s(z) \Longrightarrow z = f(x)$.*

Proof. By definition of f^*, $\text{pro } \boldsymbol{z} = \text{range}\,(f, \text{pro } \boldsymbol{x})$. As f is continuous and strictly monotone, it is an homeomorphism between $\text{pro } \boldsymbol{x}$ and $\boldsymbol{z}$. Therefore, it inverse $f^{-1} : \text{pro } \boldsymbol{z} \longrightarrow \text{pro } \boldsymbol{x}$ is also continuous. Finally, $x = f^{-1}(z) \Longrightarrow z = f(x)$ so f^{-1} (restricted to $\text{pro } \boldsymbol{x}$) is the wanted continuous function. $\square$

Finally, the next proposition stands for $\sin x$, $\cos x$, $\text{abs}\,x$, x^n for n even.

Proposition C.3. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous and $\boldsymbol{x} \in \overline{\mathbb{IR}}$. Suppose that f has no local maximum or minimum in $\text{pro } \boldsymbol{x}$ excepted maybe on the bounds of $\text{pro } \boldsymbol{x}$ and that f is strictly monotonic between its different minimum and maximum (local or global). Then there exists a* **continuous** *function $s : \text{pro } \boldsymbol{z} \longrightarrow \text{pro } \boldsymbol{x}$ which satisfies $x = s(z) \Longrightarrow z = f(x)$.*

Proof. Pick up a global minimum $x_1 \in \text{pro } \boldsymbol{x}$ and a global maximum $x_2 \in \text{pro } \boldsymbol{x}$ which are not separated by any minimum or maximum. This is possible thanks to the hypothesis that there are no local extrema in the interior of $\text{pro } \boldsymbol{x}$. Define $\boldsymbol{x}' = x_1 \vee x_2$. The function $f|_{\boldsymbol{x}'}$ is either strictly increasing or strictly decreasing, so the previous proposition applies. Finally, $s(x) \in \boldsymbol{x}' \subseteq \boldsymbol{x}$ and $x = s(z) \Longrightarrow z = f|_{\boldsymbol{x}'}(x) = f(x)$. $\square$

Two variables elementary functions

In the case of two variables elementary functions, the case where $\boldsymbol{x} \in \mathbb{IR}$ and $\boldsymbol{y} \in \mathbb{IR}$ is already treated. First of all, only $+$ and $\times$ are now considered because

$$\boldsymbol{x} - \boldsymbol{y} = \boldsymbol{x} + (-\boldsymbol{y}) \ \text{ et } \ \boldsymbol{x}/\boldsymbol{y} = \boldsymbol{x} \times (1/\boldsymbol{y})$$

The next proposition stands for all the two variable elementary functions and for $\boldsymbol{x} \in \overline{\mathbb{IR}}$ and $\boldsymbol{y} \in \overline{\mathbb{IR}}$. In the sequel, we suppose that pro $\boldsymbol{x}_1$ and pro $\boldsymbol{x}_2$ are not degenerate, the other cases being similar and simpler.

Proposition C.4. *Let $\boldsymbol{x}_1 \in \overline{\mathbb{IR}}$, $\boldsymbol{x}_2 \in \overline{\mathbb{IR}}$ and $\boldsymbol{x}_0 = \boldsymbol{x}_1 \circ \boldsymbol{x}_2$ for $\circ \in \{+, \times\}$. Then $\boldsymbol{x}_0$ is also improper and there exists a **continuous** function $s : \boldsymbol{x}_0 \longrightarrow$ pro $\boldsymbol{x}_{\{1,2\}}$ which satisfies $x_{\{1,2\}} = s(x_0) \Longrightarrow x_0 = f(x_1, x_2)$ where $f(x_1, x_2) = x_1 \circ x_2$.*

Proof. $\boldsymbol{x}_0$ is improper because the result of Kaucher arithmetic operations applied to improper intervals is also improper. Define $\boldsymbol{x}_1' = \text{pro } \boldsymbol{x}_1$, $\boldsymbol{x}_2' = \text{pro } \boldsymbol{x}_2$ and $\boldsymbol{x}_0' = \text{pro } \boldsymbol{x}_0$. Using the expressions of Kaucher arithmetic, we have $\boldsymbol{x}_0' = \text{range}\,(f, \boldsymbol{x}_1', \boldsymbol{x}_2')$. So, $\boldsymbol{x}_0' = [u_1 \circ u_2, v_1 \circ v_2]$ with $u_{\{1,2\}} \in \boldsymbol{x}_{\{1,2\}}'$ and $v_{\{1,2\}} \in \boldsymbol{x}_{\{1,2\}}'$ ($u_1 \circ u_2 < v_1 \circ v_2$ because $\boldsymbol{x}_0'$ is not degenerate by hypothesis). Now define the continuous function $m_{\{1,2\}} : [0,1] \longrightarrow \boldsymbol{x}_{\{1,2\}}'$ by

$$m_{\{1,2\}}(t) = u_{\{1,2\}}(1-t) + v_{\{1,2\}}t$$

and the continuous function $g : [0,1] \longrightarrow \boldsymbol{x}_0'$ by

$$g(t) = f(m_{\{1,2\}}(t)) = \big(u_1(1-t) + v_1 t\big) \circ \big(u_2(1-t) + v_2 t\big)$$

We have $g(0) = \inf \boldsymbol{x}_0'$ and $g(1) = \sup \boldsymbol{x}_0'$ so $\text{range}\,(g, [0,1]) = \boldsymbol{x}_0'$. We now prove that the propositions C.2 and C.3 applies to g whatever is $\circ$:

$\circ = +$ We have $g(t) = (u_1 + u_2)(1-t) + (v_1 + v_2)t$ which is strictly increasing because we supposed $u_1 + u_2 < v_1 + v_2$. So C.2 applies.

$\circ = \times$ We have $g(t) = at^2 + bt + c$. If $a \neq 0$ the Proposition C.3 applies. If $a = 0$ and $b \neq 0$ then Proposition C.2 applies. We cannot have $a = 0$ and $b = 0$ since $\boldsymbol{x}_0'$ is supposed not to be degenerate.

Therefore, there exists a continuous function $\tilde{s} : \boldsymbol{x}_0' \longrightarrow [0,1]$ which satisfies

$$t = \tilde{s}(x_0) \Longrightarrow x_0 = g(t). \tag{24}$$

It eventually easy to check that the wanted continuous function is $s(x_0) = m_{\{1,2\}}(\tilde{s}(x_0))$: on one hand $s(x_0) \in \boldsymbol{x}_{\{1,2\}}'$ by construction of $m_{\{1,2\}}$. On the other hand, $x_{\{1,2\}} = s(x_0)$ implies $x_{\{1,2\}} = m_{\{1,2\}}(\tilde{s}(x_0))$ which implies $f(x_{\{1,2\}}) = f(m_{\{1,2\}}(\tilde{s}(x_0))) = g(\tilde{s}(x_0))$. This latter is equal to x_0 thanks to (24). $\square$

It remains to study the cases where $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ do not have the same proper/improper quality. The next proposition stands for Kaucher addition.

Proposition C.5. *Let $\boldsymbol{x}_1 \in \mathbb{KR}$, $\boldsymbol{x}_2 \in \mathbb{KR}$ and $\boldsymbol{x}_0 = \boldsymbol{x}_1 + \boldsymbol{x}_2$. Define the following sets of indices: $\mathcal{A} = \mathcal{P}(\boldsymbol{x}_{\{1,2\}}) \bigcup \mathcal{I}(\boldsymbol{x}_0)$ and $\mathcal{E} = \mathcal{I}(\boldsymbol{x}_{\{1,2\}}) \bigcup \mathcal{P}(\boldsymbol{x}_0)$ ($\mathcal{A}$ contains the indices of the universally quantified variable and $\mathcal{E}$ contains the indices of the existentially quantified ones). Then both $\mathcal{A}$ and $\mathcal{E}$ are nonempty and there exists a **continuous** function $s : \boldsymbol{x}_{\mathcal{A}} \longrightarrow \boldsymbol{x}_{\mathcal{E}}$ which satisfies $x_{\mathcal{E}} = s(x_{\mathcal{A}}) \Longrightarrow x_0 = x_1 + x_2$.*

Proof. We have only two cases to study because $\boldsymbol{x}_1 + \boldsymbol{x}_2 = \boldsymbol{x}_2 + \boldsymbol{x}_1$.

(1) $\boldsymbol{x}_1 \in \mathbb{IR}$, $\boldsymbol{x}_2 \in \overline{\mathbb{IR}}$ and $\boldsymbol{x}_0 \in \mathbb{IR}$. We have to prove the existence of a continuous function $s_{\{0,2\}} : \boldsymbol{x}_1 \longrightarrow$ pro $\boldsymbol{x}_{\{0,2\}}$ which satisfies $x_{\{0,2\}} = s_{\{0,2\}}(x_1) \Longrightarrow x_0 = x_1 + x_2$. Thanks to the rules of Kaucher arithmetic, we have $\boldsymbol{x}_0 = \boldsymbol{x}_1 + \boldsymbol{x}_2 \iff (\text{dual } \boldsymbol{x}_1) = (\text{dual } \boldsymbol{x}_0) - \boldsymbol{x}_2$. Then, we can apply the Proposition C.4 to this latter operation, which proves the existence of a continuous function $s_{\{0,2\}} : \boldsymbol{x}_1 \longrightarrow$ pro $\boldsymbol{x}_{\{0,2\}}$ which satisfies $x_{\{0,2\}} = s_{\{0,2\}}(x_1) \Longrightarrow x_1 = x_0 - x_2$. Therefore, as $x_1 = x_0 - x_2 \iff x_0 = x_1 + x_2$, $s_{\{0,2\}}$ is the wanted continuous function.

(2) $\boldsymbol{x}_1 \in \mathbb{IR}$, $\boldsymbol{x}_2 \in \overline{\mathbb{IR}}$ and $\boldsymbol{x}_0 \in \overline{\mathbb{IR}}$. We have to prove the existence of a continuous function $s_2 :$ pro $\boldsymbol{x}_{\{0,1\}} \longrightarrow$ pro $\boldsymbol{x}_2$ which satisfies $x_2 = s_2(x_0, x_1) \Longrightarrow x_0 = x_1 + x_2$. It is proved in the same way than previously, noticing that $\boldsymbol{x}_0 = \boldsymbol{x}_1 + \boldsymbol{x}_2$ is equivalent to $(\text{dual } \boldsymbol{x}_2) = (\text{dual } \boldsymbol{x}_0) - \boldsymbol{x}_1$. All the intervals being proper in this latter expression, we can apply Proposition C.1. $\square$

Now, Kaucher multiplication is studied.

Proposition C.6. *Let $\boldsymbol{x}_1 \in \mathbb{KR}$, $\boldsymbol{x}_2 \in \mathbb{KR}$ and $\boldsymbol{x}_0 = \boldsymbol{x}_1\boldsymbol{x}_2$. Define the following sets of indices: $\mathcal{A} = \mathcal{P}(\boldsymbol{x}_{\{1,2\}}) \bigcup \mathcal{I}(\boldsymbol{x}_0)$ and $\mathcal{E} = \mathcal{I}(\boldsymbol{x}_{\{1,2\}}) \bigcup \mathcal{P}(\boldsymbol{x}_0)$ ($\mathcal{A}$ contains the indices of the universally quantified variable and $\mathcal{E}$ contains the indices of the existentially quantified ones). Then both $\mathcal{A}$ and $\mathcal{E}$ are nonempty and there exists a* **continuous** *function $s : \boldsymbol{x}_\mathcal{A} \longrightarrow \boldsymbol{x}_\mathcal{E}$ which satisfies $x_\mathcal{E} = s(x_\mathcal{A}) \Longrightarrow x_0 = x_1x_2$.*

Proof. Define $\boldsymbol{x}_1' = $ pro $\boldsymbol{x}_1$, $\boldsymbol{x}_2' = $ pro $\boldsymbol{x}_2$ and $\boldsymbol{x}_0' = $ pro $\boldsymbol{x}_0$. First, if the involved intervals do not contain 0 we can come back to Kaucher addition using the exp and ln bijections:

(1) $\boldsymbol{x}_1 > 0$ and $\boldsymbol{x}_2 > 0$. Apply ln to $\boldsymbol{x}_0 = \boldsymbol{x}_1\boldsymbol{x}_2$ in order to obtain the equivalent expression $\ln \boldsymbol{x}_0 = \ln \boldsymbol{x}_1 + \ln \boldsymbol{x}_2$ which is written $\tilde{\boldsymbol{x}}_0 = \tilde{\boldsymbol{x}}_1 + \tilde{\boldsymbol{x}}_2$ with $\tilde{\boldsymbol{x}}_k = \ln \boldsymbol{x}_k$. Then, the wanted continuous function is built using the one obtained thanks to the application of the Proposition C.5 to $\tilde{\boldsymbol{x}}_0 = \tilde{\boldsymbol{x}}_1 + \tilde{\boldsymbol{x}}_2$: the function ln keeps unchanged the proper/improper quality of the intervals, so the Proposition C.5 proves the existence of a continuous function $\tilde{s} :$ pro $\tilde{\boldsymbol{x}}_\mathcal{A} \longrightarrow$ pro $\tilde{\boldsymbol{x}}_\mathcal{E}$ which satisfies $\tilde{x}_\mathcal{E} = \tilde{s}(\tilde{x}_\mathcal{A}) \Longrightarrow \tilde{x}_0 = \tilde{x}_1 + \tilde{x}_2$. Then define $s(x_\mathcal{A}) = \exp \tilde{s}(\ln x_\mathcal{A})$ (here the functions ln and exp are applied to vectors componentwise). One one hand, we have pro $\ln \boldsymbol{x}_k = \ln$ pro $\boldsymbol{x}_k$, so s is defined inside $\boldsymbol{x}_\mathcal{A}'$ with values inside $\boldsymbol{x}_\mathcal{E}'$, i.e. $s : \boldsymbol{x}_\mathcal{A}' \longrightarrow \boldsymbol{x}_\mathcal{E}'$. On the other hand, using the definitions and properties of s and $\tilde{s}$, we have $x_\mathcal{E} = s(x_\mathcal{A}) \Longrightarrow x_\mathcal{E} = \exp \tilde{s}(\ln x_\mathcal{A}) \Longrightarrow \ln x_\mathcal{E} = \tilde{s}(\ln x_\mathcal{A}) \Longrightarrow \ln x_0 = \ln x_1 + \ln x_2 \Longrightarrow x_0 = x_1x_2$.

(1') $0 \notin \boldsymbol{x}_1'$ and $0 \notin \boldsymbol{x}_2'$. We come back to the first case in the following way: for $k \in \{1,2\}$ define $\epsilon_k \in \{-1,1\}$ such that $\epsilon_k x_k > 0$. So $\boldsymbol{x}_1\boldsymbol{x}_2 = (\epsilon_1\epsilon_2)(\epsilon_1\boldsymbol{x}_1)(\epsilon_2\boldsymbol{x}_2)$.

(2-4) It remains to study only three cases where $\boldsymbol{x}_1 \in \mathbb{IR}$ and $\boldsymbol{x}_2 \in \overline{\mathbb{IR}}$, because $\boldsymbol{x}_0\boldsymbol{x}_1 = \boldsymbol{x}_1\boldsymbol{x}_0$.

(2) $0 \in \boldsymbol{x}_1'$ and $0 \notin \boldsymbol{x}_2'$ which imply $\boldsymbol{x}_0 \in \mathbb{IR}$. The existence of the continuous function s is proved in a similar way to the case (1) of the Proposition C.5, considering that $\boldsymbol{x}_0 = \boldsymbol{x}_1\boldsymbol{x}_2 \iff (\text{dual } \boldsymbol{x}_1) = (\text{dual } \boldsymbol{x}_0)/\boldsymbol{x}_2$, which is well defined because $0 \notin \boldsymbol{x}_2'$. All intervals are improper in the latter expression, so we can apply Proposition C.4 in order to get the wanted continuous function.

(3) $0 \notin \boldsymbol{x}_1'$ and $0 \in \boldsymbol{x}_2'$ which imply $\boldsymbol{x}_0 \in \overline{\mathbb{IR}}$. In the same way as previously, we consider the equivalent expression $(\text{dual } \boldsymbol{x}_2) = (\text{dual } \boldsymbol{x}_0)/\boldsymbol{x}_1$ where all intervals are proper. So we can apply Proposition C.1 in order to get the wanted continuous function.

(4) $0 \in \boldsymbol{x}_1'$ and $0 \in \boldsymbol{x}_2'$. In this case, $\boldsymbol{x}_0 = [0,0]$, and the wanted continuous function is for example $s_2(x_1, x_0) = 0 \in \boldsymbol{x}_2'$ as $x_2 = s(x_1, x_0) \Longrightarrow x_1 \times x_2 = x_0$. $\square$

References

[1] G. Alefeld and J. Herzberger. *Introduction to Interval Computations.* Computer Science and Applied Mathematics, 1974.

[2] J. Armengol, J. Vehí, L. Travé-Massuyès, and M.A. Sainz. Application of modal intervals to the generation of error-bounded envelopes. *Reliab. Comp.*, 7(2):171–185, 2001.

[3] F. Benhamou and F. Goualard. Universally quantified interval constraints. In *Proceedings of International Conference on Principles and Practice of Constraint Programming*, volume 1894 of *LNCS*, pages 67–82, 2000.

[4] H. Collavizza, F. Delobel, and M. Rueher. Comparing Partial Consistencies. *Reliab. Comp.*, 1:1–16, 1999.

[5] N.S. Dimitrova, S.M. Markov, and E.D. Popova. Extended interval aritrhmetics : New results and applications. In L. Atanassova and J. Herzberger, editors, *Computer Arithmetics and Enclosure Methods*, pages 225–232. Elsevier Sci. Publishers, 1992.

[6] E. Gardenyes, H. Mielgo, and A. Trepat. Modal Intervals: Reason and Ground Semantics. *Interval Mathematics, LNCS*, 212:27–35, 1985.

[7] A. Goldsztejn. *Définition et Applications des Extensions des Fonctions Réelles aux Intervalles Généralisés.* PhD thesis, Université de Nice-Sophia Antipolis, 2005.

[8] B. Hayes. A Lucid Interval. *American Scientist*, 91(6):484–488, 2003.

[9] P. Herrero, M.A. Sainz, J. Vehí, and L. Jaulin. Quantified Set Inversion with Applications to Control. In *IEEE International Symposium on Computer Aided Control Systems Design*, 2004.

[10] P. Herrero, M.A. Sainz, J. Vehí, and L. Jaulin. Quantified Set Inversion Algorithm with Applications to Control. In *Proc. of Interval Mathematics and Constrained Propagation Methods*, volume 11(5) of *Reliab. Comp.*, 2005.

[11] V.I. Istratescu. *Fixed point theory : an introduction.* Mathematics and its applications. D. Reidel, 1981.

[12] L. Jaulin, M. Kieffer, O. Didrit, and E. Walter. *Applied Interval Analysis with Examples in Parameter and State Estimation, Robust Control and Robotics.* Springer-Verlag, 2001.

[13] E. Kaucher. *Uber metrische und algebraische Eigenschaften einiger beim numerischen Rechnen auftretender Raume.* PhD thesis, Karlsruhe, 1973.

[14] E. Kaucher. Interval Analysis in the Extended Interval Space $\mathbb{IR}$. *Computing*, Suppl. 2:33–49, 1980.

[15] R. B. Kearfott. Interval Arithmetic Techniques in the Computational Solution of Nonlinear Systems of Equations: Introduction, Examples, and Comparisons. *Computational Solution of Nonlinear Systems of Equations, Amer. Math. Soc.*, 26:337–358, 1990.

[16] R. B. Kearfott. Interval Computations: Introduction, Uses, and Resources. *Euromath*, Bulletin 2(1):95–112, 1996.

[17] R. B. Kearfott. Interval Analysis : Interval Fixed Point Theory. In *Encyclopedia of Optimization*, volume 3, pages 48–51. Kluwer, 2001.

[18] R. B. Kearfott. Interval Newton Methods. In *Encyclopedia of Optimization*, volume 3, pages 76–78. Kluwer, 2001.

[19] R. B. Kearfott, M. T. Nakao, A. Neumaier, S. M. Rump, S. P. Shary, and P. Van Hentenryck. Standardized Notation in Interval Analysis. 2002.

[20] V. Kreinovich, A. Lakeyev, J. Rohn, and P. Kahl. *Computational Complexity and Feasibility of Data Processing and Interval Computations*. Kluwer Academic Publishers, 1998.

[21] S. Markov. On Directed Interval Arithmetic and its Applications. *Journal of Universal Computer Science*, 1(7):514–526, 1995.

[22] R. Moore. *Interval Analysis*. Prentice-Hall, 1966.

[23] R.E. Moore. A test for existence of solutions to nonlinear systems. *SIAM J. Numer. Anal.*, 14:611–626, 1977.

[24] R.E. Moore and L. Qi. A successive interval test for nonlinear systems. *SIAM J. Numer. Anal.*, 19:845–895, 1982.

[25] A. Neumaier. *Interval Methods for Systems of Equations*. Cambridge Univ. Press, 1990.

[26] H.-J. Ortolf. Eine Verallgemeinerung der Intervallarithmetik. *Geselschaft fuer Mathematik und Datenverarbeitung, Bonn*, 11:1–71, 1969.

[27] H. Ratschek and J. Rokne. *Computer Methods for the Range of Functions*. Ellis Horwood Publ., 1984.

[28] S. M. Rump. Computer-Assisted Proofs and Self-Validating Methods. In B. Einarsson, editor, *Accuracy and Reliability in Scientific Computation, SIAM*, pages 195–240. 2005.

[29] M.A. Sainz, J. Armengol, and J. Vehí. Fault Detection and Isolation of the Three-Tank System Using Modal Interval Analysis. *Journal of Process Control*, 12(2):325–338, 2002.

[30] M.A. Sainz, E. Gardenyes, and L. Jorba. Formal Solution to Systems of Interval Linear or Non-Linear Equations. *Reliab. Comp.*, 8(3):189–211, 2002.

[31] M.A. Sainz, E. Gardenyes, and L. Jorba. Interval Estimations of Solutions Sets to Real-Valued Systems of Linear or Non-Linear Equations. *Reliab. Comp.*, 8(4):283–305, 2002.

[32] S.P. Shary. Algebraic Approach to the Interval Linear Static Identification, Tolerance and Control Problems, or One More Application of Kaucher Arithmetic. *Reliab. Comp.*, 2:3–33, 1996.

[33] S.P. Shary. A new technique in systems analysis under interval uncertainty and ambiguity. *Reliab. Comp.*, 8:321–418, 2002.

[34] SIGLA/X group. Modal intervals (basic tutorial). *Applications of Interval Analysis to Systems and Control (Proceedings of MISC'99)*, pages 157–227, 1999.

[35] SIGLA/X group. Modal intervals. *Reliab. Comp.*, 7:77–111, 2001.

[36] T. Sunaga. Theory of interval algebra and its application to numerical analysis. *In RAAG Memoirs, Ggujutsu Bunken Fukuy-kai*, 2:29–46, 1958.

[37] J. Vehí, M.A. Sainz, J. Armengol, and I. Ferrer. Applications of modal interval analysis to systems and control. In *Current trends in Qualitative Reasoning and Applications*, pages 49–64. Universidad de Sevilla, 2000.

[38] M. Warmus. Calculus of Approximations. *Bulletin de l'Académie Polonaise des Sciences*, 4(5):253–257, 1956.

[39] Manna Z. and Waldinger R. *The Deductive Foundations of Computer Programming*. AddisonWesley, 1993.