On Boundedness and Unboundedness of Polyhedral Estimates for Reachable Sets of Linear Differential Systems*

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Abstract

The paper describes the properties of boundedness and unboundedness of outer polyhedral (parallelepiped-valued) estimates for reachable sets of linear differential systems with stable matrices over the infinite time interval. New results concerning tight estimates are presented.

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1 Introduction

The problem of constructing trajectory tubes (in particular, reachable tubes which describe a dynamic of reachable sets) is an essential theme in control theory [14]. Since practical construction of these tubes may be cumbersome, different numerical methods have been devised for this purpose. Among them are techniques developed for estimating reachable sets by domains of some fixed shape such as ellipsoids, parallelepipeds, and zonotopes (see, for example, [1, 3, 5, 7–9, 12–15, 21] and references therein). In particular, box-valued estimates may be constructed by means of interval calculations (see, for example, [7]), but such estimates can be rather conservative and even unbounded due to the well-known wrapping effect [4, 17, 19]. Several ideas have been proposed to reduce this effect (see, for example, [18] and references therein). In particular, Tucker [22] proposed a method based on a partitioning process and using

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sub-pavings of interval vectors. Tucker's method may require extensive computation and memory for large dimensional systems.

To make representations of reachable sets as nearly exact as possible, A.B. Kurzhanski proposed using families of fixed shape estimates [12, 14, 15], especially families of tight estimates [15]. We expanded this approach to polyhedral (parallelepiped-valued) estimates. The family $\mathfrak{P}$ of outer polyhedral estimates of reachable sets for linear differential systems with parallelepiped-valued uncertainties in initial states and additive inputs was introduced [9]. These estimates are determined by a given dynamics of orientation matrices $P(t) \in \mathbb{R}^{n\times n}$ (this function is the parameter of the family) and by corresponding parametrized differential equations, which describe the dynamics of centers and "semi-axis" values of parallelepipeds. Considering different types of the orientation matrix dynamics $P(\cdot)$, we obtain several subfamilies $\mathfrak{P}^i \in \mathfrak{P}$ of the estimates with different properties, in particular, subfamilies $\mathfrak{P}^3$ and $\mathfrak{P}^1$ of tight and touching [8] (tight in n specific directions) estimates (both ensure the exact representations of reachable sets through intersections of their units). Box-valued estimates may be attributed to the subfamily $\mathfrak{P}^2$ of estimates with constant orientation matrices. In fact, the orientation matrix $V = P(0)$ at the initial time is the parameter of all subfamilies $\mathfrak{P}^i$.

The paper presents our recent results on studying the properties of boundedness and unboundedness on the infinite time interval of outer polyhedral estimates for reachable sets of linear differential systems with constant stable matrices. The properties mentioned are determined by interactions of three factors: the matrix V, the real Jordan matrix for system's matrix, and the properties of the bounding sets for uncertainties. The results of this interaction are different for different subfamilies $\mathfrak{P}^i$. We recall some of the corresponding criteria [10, 11] for boundedness / unboundedness of estimates from $\mathfrak{P}^1$ and $\mathfrak{P}^2$, including characterizing the possible degree of the growth of the estimates in terms of the exponents. Then we present new results concerning the subfamily $\mathfrak{P}^3$ of tight estimates. In particular, for two-dimensional systems, all estimates from $\mathfrak{P}^3$ are bounded, and they are orthogonal parallelepipeds. The results of numerical simulations are presented.

We use the following notation: $\mathbb{R}^n$ is the n-dimensional vector space; $\top$ is the transposition symbol; $\|x\|_2 = (x^\top x)^{1/2}$, $\|x\|_1 = \sum_{i=1}^n |x_i|$, $\|x\|_\infty = \max_{1\le i\le n} |x_i|$ are vector norms for $x{=}(x_1, \dots, x_n)^\top \in \mathbb{R}^n$; the symbol $\|x\|$ without a subscript means $\|x\| = \|x\|_2$; $\mathrm{e}^i{=}(0, \dots, 0, 1, 0, \dots, 0)^\top$ is the unit vector oriented along the axis x_i (the unit stands at position i); $\mathrm{e}{=}(1, 1, \dots, 1)^\top$; $\mathbb{R}^{n\times m}$ is the space of real $n{\times}m$-matrices $A = \{a_i^j\} = \{a^j\} = \{a^1, \dots, a^m\}$ with elements a_i^j and columns a^j (the upper index numbers the columns and the lower index numbers the components of vectors); I is the identity matrix; 0 is the zero matrix (vector); $\mathrm{Abs}\, A{=}\{|a_i^j|\}$ for $A = \{a_i^j\} \in \mathbb{R}^{n\times m}$; $\mathrm{Ab}\, A$ for $A{\in}\mathbb{R}^{n\times n}$ is the matrix such that $(\mathrm{Ab}\, A)_i^i = a_i^i$, $(\mathrm{Ab}\, A)_i^j = |a_i^j|$, $i \neq j$; and the notation $k = 1, \dots, N$ is used instead of $k = 1, 2, \dots, N$ for brevity.

2 Problems

Let $x \in \mathbb{R}^n$ denote the state. Consider the system

$$\dot{x} = A(t)\, x + w(t), \quad t \in \mathcal{T} = [0, \theta]. \tag{1}$$

The initial state $x(0) = x_0 \in \mathbb{R}^n$ and the input (control/disturbance) $w(t) \in \mathbb{R}^n$ (assumed to be a Lebesgue measurable function) are unknown but subjected to set-valued constraints

$$x_0 \in \mathcal{X}_0, \quad w(t) \in \mathcal{R}(t), \quad \text{a.e.} \quad t \in \mathcal{T}, \tag{2}$$

where $\mathcal{X}_0$, $\mathcal{R}(t)$ are given convex compact sets in $\mathbb{R}^n$, and the set-valued map $\mathcal{R}(t)$ is continuous.

First, recall some definitions that will be used below.

Let $\mathcal{X}(t) = \mathcal{X}(t, 0, \mathcal{X}_0)$ be a *reachable set* of the system (1), (2) at time $t > 0$, that is, the set of all points $x \in \mathbb{R}^n$, for each of which there exist x_0 and $w(\cdot)$ that satisfy (2) and generate a solution $x(\cdot)$ of (1) that satisfies $x(t) = x$. The multivalued function $\mathcal{X}(t)$, $t \in \mathcal{T}$, is known as a *trajectory* (or *reachable*) *tube* $\mathcal{X}(\cdot)$.

By a *parallelepiped* $\mathcal{P}(p, P, \pi) \subset \mathbb{R}^n$, we mean a set such that $\mathcal{P} = \mathcal{P}(p, P, \pi) = \{x \in \mathbb{R}^n |\ x = p + \sum_{i=1}^n p^i \pi_i \xi_i,\ \mathrm{Abs}\, \xi \le \mathrm{e}\}$, where $p \in \mathbb{R}^n$; $P = \{p^i\} \in \mathbb{R}^{n \times n}$ is a nonsingular matrix ($\det P \ne 0$) such that $\|p^i\| = 1$; $\pi \in \mathbb{R}^n$, $\pi \ge 0$, $\mathrm{e} = (1, \ldots, 1)^\top$. Vector inequalities $(\le, <, \ge, >)$ are understood component-wise; the condition $\|p^i\| = 1$ may be omitted to simplify formulas. It may be said that p determines the center of the parallelepiped, P is the orientation matrix, p^i are the "directions", and π_i are the values of its "semi-axes".

We call $\mathcal{P}$ an *outer estimate for* $\mathcal{X} \subset \mathbb{R}^n$ if $\mathcal{P} \supseteq \mathcal{X}$.

An outer estimate $\mathcal{P}$ of $\mathcal{X}$ is *tight (in direction l)* [15] if $\rho(\pm l|\mathcal{P}) = \rho(\pm l|\mathcal{X})$, where $\rho(l|\mathcal{X}) = \sup\{l^\top x \,|\, x \in \mathcal{X}\}$ stands for the support function of a set $\mathcal{X} \subset \mathbb{R}^n$ at $l \in \mathbb{R}^n$.

A parallelepiped-valued outer estimate $\mathcal{P}(p, P, \pi)$ for $\mathcal{X}$ is *touching* if it is a tight estimate in n specified directions $l^i = {P^{-1}}^\top \mathrm{e}^i$, $i = 1, \ldots, n$.

Everywhere below we accept the following

Assumption 1 The sets $\mathcal{X}_0$ and $\mathcal{R}(t)$ are parallelepipeds:

$$\mathcal{X}_0 = \mathcal{P}_0 = \mathcal{P}(p_0, P_0, \pi_0), \quad \mathcal{R}(t) = \mathcal{P}(r(t), R(t), \rho(t)), \tag{3}$$

where $r(\cdot)$, $\rho(\cdot)$ and $R(\cdot)$ are continuous vector and matrix functions.

The definition of a parallelepiped implies its support function satisfies $\rho(l|\mathcal{P}(p, P, \pi)) = l^\top p + \mathrm{Abs}\,(l^\top P)\pi$.

It is known that the reachable sets of the system (1), (2), (3) are determined by the multivalued Aumann integral

$$\mathcal{X}(t) = p(t) + \Phi(t, 0)(\mathcal{P}_0 - p_0) + \int_0^t \Phi(t, \tau)(\mathcal{R}(\tau) - r(\tau))\, d\tau; \tag{4}$$

the support function of $\mathcal{X}(t)$ has the form

$$\rho(l|\mathcal{X}(t)) = l^\top p(t) + \mathrm{Abs}\,(l^\top \Phi(t, 0) P_0)\, \pi_0 + \int_0^t \mathrm{Abs}\,(l^\top \Phi(t, \tau) R(\tau))\, \rho(\tau)\, d\tau. \tag{5}$$

Here, $p(t)$ is determined by the differential system

$$\dot{p} = A(t)\, p + r(t), \quad p(0) = p_0, \tag{6}$$

$\Phi(t, \tau) = \Phi(t)\Phi^{-1}(\tau)$ is the Cauchy matrix, where $\Phi(t)$ is the fundamental matrix solution of $\dot{x} = A(t)x$ satisfying $\dot{\Phi} = A(t)\Phi$, $\Phi(0) = I$. If A is a constant matrix, then $\Phi(t) = e^{At}$ and the following estimates are valid for any $\varepsilon > 0$ [2, p.57] (where $C = \mathrm{Const}\,(\varepsilon) > 0$):

$$\|e^{At}\| \le Ce^{(-\mathrm{m}+\varepsilon)t},\ \|e^{-At}\| \le Ce^{(\mathrm{M}+\varepsilon)t},\ t \in [0, \infty),\ \mathrm{m} = \min|\mathrm{Re}\,\lambda_k|,\ \mathrm{M} = \max|\mathrm{Re}\,\lambda_k|, \tag{7}$$

where $\lambda_k = \operatorname{Re}\lambda_k + \operatorname{Im}\lambda_k \cdot \sqrt{-1}$ for $k = 1, \dots, n$ are eigenvalues of the matrix $A \in \mathbb{R}^{n\times n}$ (the estimates can be improved for a diagonalizable matrix A by setting $\varepsilon = 0$).

Thus, the sets $\mathcal{X}(t)$ are not obliged to be parallelepipeds under Assumption 1. We investigate the possibilities of the outer polyhedral estimation of the tubes $\mathcal{X}(\cdot)$.

In [9], there is a description of a parametrized family $\mathfrak{P}$ of parallelepiped-valued estimates $\mathcal{P}(t) = \mathcal{P}(p(t), P(t), \pi(t))$, $t \in \mathcal{T}$, that are outer for reachable set $\mathcal{X}(t)$ and possess the evolutionary properties [8] (the"upper" semigroup property [14] and the superreachability property [1]), which are analogs of the semigroup property [1, 14] inherent in reachable sets $\mathcal{X}(t)$. The parameter of the family is an arbitrary continuously differentiable function $P(t) \in \mathbb{R}^{n\times n}$ with $\det P(t) \neq 0$, $t \in \mathcal{T}$, which specifies the dynamics of the orientation matrices. The functions $p(\cdot)$ and $\pi(\cdot)$ are determined by the equations (6), and

$$\dot{\pi} = \operatorname{Ab}(P^{-1}(AP - \dot{P}))\pi + \operatorname{Abs}(P^{-1}R)\rho, \quad \pi(0) = \operatorname{Abs}(P(0)^{-1}P_0)\pi_0, \tag{8}$$

where $\operatorname{Abs} B$ and $\operatorname{Ab} B$ stand for the operations of replacing all the elements and all the off-diagonal elements of B, respectively, by their absolute values.

Choosing different types of the orientation matrix dynamics $P(\cdot)$, we obtain estimates with different properties. Let us recall two subfamilies of estimates we considered earlier.

Let $\mathfrak{P}^1$ be a subfamily of $\mathfrak{P}$ for which the functions $P(\cdot)$ satisfy $\dot{P} = AP$, $P(0) = V$ (the nonsingular matrix V is a parameter of $\mathfrak{P}^1$). Such estimates are touching ones and ensure exact representations for $\mathcal{X}(t)$ through the intersections of estimates (see [8]).

If $P(t) \equiv I$, then the above equations give coordinate-wise estimates of the reachable sets in the form of boxes (interval vectors), which are used as estimating sets in classical interval analysis. Note that in the special case ($P(t) \equiv I$, $\mathcal{X}_0$ and $\mathcal{R}(t)$ are boxes centered at the origin), these estimates become known ones (see [21]).

Let $\mathfrak{P}^2 \subset \mathfrak{P}$ denote a subfamily of estimates with constant orientation matrices $P(t) \equiv V$ (evidently, including coordinate-wise estimates).

In [10, 11], the issues related to the boundedness or unboundedness on the interval $\mathcal{T} = [0, \infty)$ of estimates from $\mathfrak{P}^i \in \mathfrak{P}$, $i = 1, 2$, were investigated under the following assumption, which guarantees the boundedness of the reachable sets $\mathcal{X}(t)$ themselves.

Assumption 2 The matrix $A(t) \equiv A$ is constant and stable (i.e., all its eigenvalues have negative real parts), and the mapping $\mathcal{R}(t)$, $t \in \mathcal{T} = [0, \infty]$, is bounded.

We found for which matrices V, the estimates $\mathcal{P}(\cdot)$ from the subfamilies $\mathfrak{P}^1$ and $\mathfrak{P}^2$ are bounded or unbounded; under which conditions on A, $\mathcal{P}_0$ and $\mathcal{R}(\cdot)$ there are bounded estimates in $\mathfrak{P}^1$ and $\mathfrak{P}^2$, and under which conditions there are unbounded estimates in these families; and what is a possible growth degree of the estimates, i.e., what are possible values of their exponents.

To be more precise concerning the possible growth degree, let us recall that estimates can be compared with each other with respect to a chosen criterion, namely, a functional $\mu(\mathcal{P}) = \mu(\mathcal{P}(p, P, \pi))$ satisfying the known conditions [14, p.101]. Let us introduce a vector $q = (\operatorname{Abs} P)\,\pi$ (we have $q_i = \rho(\pm e^i|\mathcal{P} - p)$). Then, a possible criterion is [10] the functional $\mu(\mathcal{P}) = \|q\|$, where $\|q\|$ is any of the three standard norms $\|q\|_\infty$, $\|q\|_1$, or $\|x\|_2$. In particular, it is convenient to use this functional for the investigation of the boundedness properties of set-valued functions $\mathcal{P}(t) = \mathcal{P}(p(t), P(t), \pi(t))$, because if $p(\cdot)$ is bounded, then the boundedness (unboundedness) of $\mathcal{P}(\cdot)$ is equivalent to the boundedness (unboundedness) of $\mu(\mathcal{P}(\cdot))$. Recall that a set-valued mapping $\mathcal{Z}(t)$, $t \in \mathcal{T}$, is called *bounded* if $\mathcal{Z}(t) \subseteq \tilde{\mathcal{Z}}$ for all $t \in \mathcal{T}$, where $\tilde{\mathcal{Z}}$ is a bounded set.

To characterize a degree of possible increasing parallelepiped-valued tubes $\mathcal{P}(\cdot)$, the exponent $\chi = \chi(\mathcal{P})$ can be used, which was introduced in [11] by analogy with [2].

By the *exponent* $\chi = \chi(\mathcal{P})$ *of the tube (estimate)* $\mathcal{P}(t)$, $t \in [0, \infty)$, we mean the characteristic exponent $\chi[\mu]$ [2, p.125] of the function $\mu(\mathcal{P}(\cdot))$, where μ is any of the three functionals mentioned above, namely, the number $\chi = \chi(\mathcal{P}) = \overline{\lim}_{t\to\infty} t^{-1} \ln \mu(\mathcal{P}(t))$ (it is independent of the norm $\|q\|$ chosen in the definition of μ).

Below, we recall (for completeness) some of results concerning the issues mentioned above for the estimates from $\mathfrak{P}^1$ and $\mathfrak{P}^2$. Further, we investigate the similar issues for one more subfamily $\mathfrak{P}^3 \subset \mathfrak{P}$ of (tight) estimates under the same Assumption 2. This is the main contribution of this paper.

3 Known Auxiliary Facts from Matrix Theory

To make the exposition self-sufficient, we recall pertinent results from matrix theory.

If A is a matrix from $\mathbb{R}^{n\times n}$, and $\lambda_k = \alpha_k + \beta_k\sqrt{-1}$, $k = 1, \ldots, m$, are all its eigenvalues with $\beta_k \geq 0$ (some of which may be equal to each other), then A can be written [2, p.465], [20, Sect. 6.6] in the form

$$
\begin{gathered}
A = T^{-1}JT, \quad \text{where} \quad J = \operatorname{diag}\{J_1, \ldots, J_m\};\\
J_k = \begin{bmatrix} S_k & I & \ldots & 0 & 0 \\ 0 & S_k & \ldots & 0 & 0 \\ & & \ldots & & \\ 0 & 0 & \ldots & S_k & I \\ 0 & 0 & \ldots & 0 & S_k \end{bmatrix} \in \mathbb{R}^{(\nu_k\gamma_k)\times(\nu_k\gamma_k)},\\
S_k, I, 0 \in \mathbb{R}^{\nu_k\times\nu_k}, \quad \nu_k = 1 \text{ or } 2, \quad k = 1, \ldots, m;\\
\nu_k = 1,\ S_k = \alpha_k,\ \text{if } \beta_k = 0; \qquad \nu_k = 2,\ S_k = \begin{bmatrix} \alpha_k & -\beta_k \\ \beta_k & \alpha_k \end{bmatrix},\ \text{if } \beta_k \neq 0,
\end{gathered}
\tag{9}
$$

where $\operatorname{diag}\{J_1, \ldots, J_m\}$ is a block-diagonal matrix with square cells (blocks) J_k on the main diagonal. A matrix $A \in \mathbb{R}^{n\times n}$ is called [16] *diagonalizable* or *simple* if $\gamma_k = 1$, $k = 1, \ldots, m$, (in other words, if it has n linearly independent complex eigenvectors) and *defective* otherwise. The matrix J is called [20] the *real Jordan matrix*. For $n = 2$, there are three different structures of J corresponding to the following cases:

case A: $\operatorname{Im}\lambda_1 = \operatorname{Im}\lambda_2 = 0$, and A is diagonalizable. Then, J $= \operatorname{diag}\{\lambda_1, \lambda_2\}$;

case B: $\operatorname{Im}\lambda_1 = \operatorname{Im}\lambda_2 = 0$, $\lambda_1 = \lambda_2 = \alpha$, and A is defective. Then, $J = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}$;

case C: $\lambda_{1,2} = \alpha \pm \beta\sqrt{-1}$ with $\beta \neq 0$. Then, $J = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$; A is a diagonalizable.

It is known [2, p. 56, 63], that the exponential of J_k in (9) is

$$
\begin{gathered}
e^{Jt} = \begin{bmatrix} \sigma_1^1 e^{St} & \sigma_1^2 t e^{St} & \ldots & \sigma_1^\gamma t^{\gamma-1} e^{St} \\ 0 & \sigma_2^2 e^{St} & \ldots & \sigma_2^\gamma t^{\gamma-2} e^{St} \\ & & \ldots & \\ 0 & 0 & \ldots & \sigma_\gamma^\gamma e^{St} \end{bmatrix} \in \mathbb{R}^{(\nu\gamma)\times(\nu\gamma)},\ \text{where } J = J_k,\ S = S_k,\ \gamma = \gamma_k,\\
\sigma_i^j = 1/(j-i)! > 0 \text{ for all } j > i,\ \sigma_i^i = 1\ (i = 1, \ldots, \gamma);\\
e^{St} = e^{\alpha t} H(t) \in \mathbb{R}^{\nu\times\nu}; \quad H = 1 \text{ if } \nu = 1; \quad H(t) = \begin{bmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{bmatrix} \text{ if } \nu = 2;\\
\alpha = \alpha_k,\ \beta = \beta_k,\ \nu = \nu_k.
\end{gathered}
\tag{10}
$$

We have the identities $H(t)^\top H(t) \equiv H(t) H(t)^\top \equiv I$, which are also useful for the future.

4 Properties of Estimates from $\mathfrak{P}^1$ and $\mathfrak{P}^2$

In this section, let Assumption 2 be satisfied. It appears, not only the estimates from $\mathfrak{P}^2$ can be unbounded on $\mathcal{T} = [0, \infty)$ (for $V = I$ this is the well-known "wrapping effect") but also touching and tight estimates from $\mathfrak{P}^1$ and $\mathfrak{P}^3$ under the following *condition of nondegeneracy for* $\mathcal{R}(\cdot)$:

$$\mathcal{R}(t) \supseteq \mathcal{P}(r(t), I, \varepsilon_0\, \mathrm{e}), \quad t \in \mathcal{T} = [0, \infty), \quad \text{where } \varepsilon_0 > 0. \tag{11}$$

Estimates from $\mathfrak{P}^2$ can be unbounded also under the *nondegeneracy condition for the initial set*:

$$\mathcal{P}_0 \supseteq \mathcal{P}(p_0, I, \varepsilon_0\, \mathrm{e}), \quad \text{where } \varepsilon_0 > 0. \tag{12}$$

The main properties of boundedness of tubes from $\mathfrak{P}^1$ are summarized in

Theorem 1 *(See [10,11]). For $\mathcal{P}(\cdot) \in \mathfrak{P}^1$, the exponent $\chi(\mathcal{P})$ satisfies $\chi(\mathcal{P}) \leq \mathrm{M}{-}\mathrm{m}$, where M and m are introduced in (7).*

If the sets $\mathcal{R}(t)$ are singletons ($\mathcal{R}(t) \equiv r(t)$), then the sets $\mathcal{P}(t)$ contract to their centers $p(t)$ (i.e., $\pi(t) \to 0$) as $t \to \infty$ for any $P(0) = V$. In the general case:

(1) If the matrix A is diagonalizable and $\mathrm{M} = \mathrm{m}$, then tubes $\mathcal{P}(\cdot)$ are bounded for any $P(0) = V$.

(2) Suppose that A is diagonalizable, $\mathrm{M} \neq \mathrm{m}$, T is the matrix from (9), and the matrices $\tilde{V} = TV$ and $\tilde{W} = V^{-1}T^{-1}$ are decomposed into the corresponding blocks $\tilde{V}_i^j \in \mathbb{R}^{\nu_i \times \nu_j}$ and $\tilde{W}_j^i \in \mathbb{R}^{\nu_j \times \nu_i}$ ($i, j{=}1, \ldots, m$) (the numeration of blocks and eigenvalues λ_i below is determined by (9)). If V is such that, for any pair of eigenvalues λ_i and λ_j satisfying the inequality $|\mathrm{Re}\, \lambda_i| < |\mathrm{Re}\, \lambda_j|$, the relation $Z_i^j = 0 \in \mathbb{R}^{\nu_i \times \nu_j}$ is valid, where $Z_i^j = \sum_{k=1}^m \mathrm{Abs}\, \tilde{V}_i^k\, \mathrm{Abs}\, \tilde{W}_k^j$, then the corresponding tube $\mathcal{P}(\cdot)$ is bounded on $\mathcal{T}$. In the case when V is such that, for some pair λ_i, λ_j with $|\mathrm{Re}\, \lambda_i| < |\mathrm{Re}\, \lambda_j|$, the relation $Z_i^j \neq 0 \in \mathbb{R}^{\nu_i \times \nu_j}$ is valid, then, under nondegeneracy condition (11), the tube $\mathcal{P}(\cdot)$ is unbounded on $\mathcal{T}$ and $\chi(\mathcal{P}) \geq |\mathrm{Re}\, \lambda_j| - |\mathrm{Re}\, \lambda_i|$. There exist matrices V that generate both bounded tubes (in particular, $V = T^{-1}$) and unbounded tubes.

(3) If A is defective and $\mathcal{R}(\cdot)$ satisfies (11), then $\mathcal{P}(\cdot)$ is unbounded for any V.

The properties of boundedness of the tubes from $\mathfrak{P}^2$ depend not only on the eigenvalues λ_k of the matrix A, but also on eigenvalues ω_k of the important auxiliary matrix $A_P = \mathrm{Ab}\,(P^{-1}AP)$, where $P = V$ is the constant orientation matrix.

Proposition 1 *(See [11]). A tube $\mathcal{P}(\cdot) \in \mathfrak{P}^2$ is bounded if one of the following two groups of conditions is satisfied:*

(1) the matrix $A_P = \mathrm{Ab}\,(P^{-1}AP)$ is stable;

(2) all $\omega_k \leq 0$, the equality $\gamma_k(\omega_k) = 1$ is satisfied for all ω_k with $\mathrm{Re}\, \omega_k = 0$, and in addition, the sets $\mathcal{R}(t)$ are singletons ($\mathcal{R}(t) \equiv r(t)$), where $\gamma_k(\omega_k)$ are defined for A_P similarly to the definition of γ_k for A in (9).

A tube $\mathcal{P}(\cdot) \in \mathfrak{P}^2$ is unbounded if one of the following two groups of conditions is satisfied:

(3) nondegeneracy condition (12) is satisfied for $\mathcal{P}_0$, and either there is an ω_k with $\mathrm{Re}\,(\omega_k) > 0$, or there is an ω_k with $\mathrm{Re}\,(\omega_k) = 0$ and $\gamma_k(\omega_k) \geq 2$;

(4) condition (11) *is satisfied for $\mathcal{R}(\cdot)$, and there exists ω_k with $\mathrm{Re}\,(\omega_k) \geq 0$.*

Proposition 2 *(See [10,11]). The following statements are true for tubes $\mathcal{P}(\cdot) \in \mathfrak{P}^2$.*

(1) If $A = \alpha I$, where $\alpha \in \mathbb{R}^1$, then the tubes $\mathcal{P}(\cdot)$ are bounded for any P.

(2) If $A \neq \alpha I$, where $\alpha \in \mathbb{R}^1$, and either $\mathcal{P}_0$ or $\mathcal{R}(\cdot)$ satisfies the nondegeneracy condition ((12) or (11)), then there are unbounded tubes $\mathcal{P}(\cdot) \in \mathfrak{P}^2$ with arbitrarily large exponents $\chi(\mathcal{P})$.

(3) If all eigenvalues λ_k of the matrix A are such that $|\mathrm{Im}\,\lambda_k| < |\mathrm{Re}\,\lambda_k|$, then there exist bounded tubes in $\mathfrak{P}^2$.

(4) If $n = 2$, case C with $|\beta| > |\alpha|$ takes place (see Sec. 3), and either $\mathcal{P}_0$ or $\mathcal{R}(\cdot)$ satisfies the nondegeneracy condition, then all tubes $\mathcal{P}(\cdot) \in \mathfrak{P}^2$ are unbounded.

For the clarity and convenience of comparing the properties of tubes from $\mathfrak{P}^1$ and $\mathfrak{P}^2$ for two-dimensional systems, we present a summary of the above results for $n = 2$ in Figure 1. Namely, for different types of the real Jordan form of the system matrix and either general or additional conditions on the bounding sets, four possible situations of boundedness and unboundedness of tubes from our subfamilies are shown[1].

Additional conditions on $\mathcal{P}_0, \mathcal{R}(\cdot)$	$\mathrm{Im}\,\lambda_1 = \mathrm{Im}\,\lambda_2 = 0$						$\lambda_{1,2} = \alpha \pm \beta\sqrt{-1}$, $\beta \neq 0$					
	A—diagonalizable: $J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$				A–defective: $J = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}$		A — diagonalizable: $J = TAT^{-1} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$					
	$\lambda_1 = \lambda_2$		$\lvert\lambda_1\rvert < \lvert\lambda_2\rvert$		$\lambda_1 = \lambda_2 = \alpha$		$\lvert\alpha\rvert > \lvert\beta\rvert$		$\lvert\alpha\rvert = \lvert\beta\rvert$		$\lvert\alpha\rvert < \lvert\beta\rvert$	
	$\mathfrak{P}^1$	$\mathfrak{P}^2$	$\mathfrak{P}^1$	$\mathfrak{P}^2$	$\mathfrak{P}^1$	$\mathfrak{P}^2$	$\mathfrak{P}^1$	$\mathfrak{P}^2$	$\mathfrak{P}^1$	$\mathfrak{P}^2$	$\mathfrak{P}^1$	$\mathfrak{P}^2$
–	🟩	🟩	🟨	🟨		🟨	🟩	🟨	🟩		🟩	
$\mathcal{R}(t) \equiv r(t)$, $\mathrm{int}\,\mathcal{P}_0 \neq \emptyset$	🟩	🟩	🟩	🟨🟥	🟩	🟨🟥	🟩	🟨🟥	🟩	🟨🟥	🟩	⬛
$\mathrm{int}\,\mathcal{R}(t) \neq \emptyset$	🟩	🟩	🟨🟥	🟨🟥	⬛	🟨🟥	🟩	🟨🟥	🟩	⬛	🟩	⬛

Figure 1: Boundedness / unboundedness of $\mathcal{P}(\cdot) \in \mathfrak{P}^i$, $i = 1, 2, 3$ (case $n = 2$). Notation: 🟩 — all tubes from $\mathfrak{P}^i$ are bounded; 🟨 — there exist bounded tubes; 🟥 — there exist unbounded tubes; ⬛ — all tubes are unbounded; $\mathrm{int}\,\mathcal{X}$ means the interior of $\mathcal{X} \subseteq \mathbb{R}^n$. It should be the case "🟩" for $\mathfrak{P}^3$ for any A, $\mathcal{P}_0$, $\mathcal{R}(\cdot)$ due to Theorem 3

5 Properties of Estimates from $\mathfrak{P}^3$

Now let us return to the system (1)-(3) and consider a subfamily $\mathfrak{P}^3 \subset \mathfrak{P}$ of tubes $\mathcal{P}(\cdot)$ such that the columns of the orientation matrices $P(t) = \{p^i(t)\}$ satisfy

$$\begin{aligned} &\dot{p}^i = A(t)p^i, \quad i = 1, \dots, n-1, \quad \dot{p}^n = -A(t)^\top p^n, \quad t \in \mathcal{T}; \\ &P(0) = \{p^i(0)\} = V = \{v^i\} \in \mathbb{R}^{n \times n}; \\ &\det V \neq 0, \quad {v^n}^\top v^i = 0, \; i = 1, \dots, n-1. \end{aligned} \tag{13}$$

The following lemma reveals the specificity of differential equations (8) for $\pi(t)$ caused by relations (13).

[1]The existence of bounded tubes $\mathcal{P}(\cdot) \in \mathfrak{P}^2$ for case C with $|\beta| = |\alpha|$ follows from Proposition 1 and [11, Lemma 2] (this lemma describes a nonempty class of "quasi-orthogonal" matrices $P \in \mathbb{R}^{2\times 2}$ for which we have $\omega_1 = 0$, $\omega_2 < 0$ in the case mentioned).

Lemma 1 *Parameters $P(\cdot)$ and $\pi(\cdot)$ of tubes $\mathcal{P}(\cdot) \in \mathfrak{P}^3$ satisfy the properties:*

(1) Columns of the orientation matrices $P(t) = \{p^i(t)\}$ satisfy the orthogonality relations

$$p^n(t)^\top p^i(t) \equiv 0, \quad t \in \mathcal{T}, \quad i = 1, \ldots, n-1, \tag{14}$$

$\det P(t) \neq 0$, $t \in \mathcal{T}$, and the last column $q^n(t)$ of the matrix $Q = (P^{-1})^\top = \{q^i\}$ is equal to

$$q^n(t) = \|p^n(t)\|^{-2} p^n(t), \quad t \in \mathcal{T}. \tag{15}$$

(2) The first $n-1$ values of semi-axes $\pi_i(t)$ can be found from the relations

$$\pi_i(t) = \tilde{\pi}_i(t), \quad t \in \mathcal{T}, \quad i = 1, \ldots, n-1, \tag{16}$$

where the n-vector function $\tilde{\pi}(t) = (\tilde{\pi}_1(t), \ldots, \tilde{\pi}_n(t))^\top \in \mathbb{R}^n$ satisfies

$$\dot{\tilde{\pi}} = \mathrm{Abs}\,(P^{-1}(A + A^\top)p^n)\,\pi_n + \mathrm{Abs}\,(P^{-1}R)\rho, \quad t \in \mathcal{T}, \quad \tilde{\pi}(0) = \pi(0). \tag{17}$$

(3) The last value $\pi_n(t)$ satisfies

$$\dot{\pi}_n = \|p^n\|^{-2} {p^n}^\top (A + A^\top) p^n\, \pi_n + \|p^n\|^{-2} \mathrm{Abs}\,({p^n}^\top R)\rho, \quad t \in \mathcal{T}, \tag{18}$$

and can be found in the following explicit form:

$$\pi_n(t) = \|p^n(t)\|^{-2} \left(\|p^n(0)\|^2 \pi_n(0) + \int_0^t \mathrm{Abs}\,(p^n(\tau)^\top R(\tau))\rho(\tau)\, d\tau \right), \quad t \in \mathcal{T}. \tag{19}$$

Proof. Identities (14) follow from the identity $p^n(t)^\top p^i(t) = {v^n}^\top v^i + \int_0^t \frac{d}{d\tau}({p^n}^\top p^i)\, d\tau$ and (13). Nonsingularity of $P(t)$ is the effect of (14) and linear independence of the vectors $p^i(t)$, $i = 1, \ldots, n{-}1$. Equality (15) follows from the definition of Q (i.e., the equality $Q^\top P = I$) and (14). Relations (16)-(18) follow from (8) and (13), taking into account that the first $n-1$ columns of the matrix $P^{-1}(AP - \dot{P})$ are zeros, and the last one is equal to $P^{-1}(A + A^\top)p^n$, and also using relation (15) for the last row of P^{-1} and the definition of the operation Ab. Equality (19) is obtained by concrete definition of the well known formula [6, Sec. 9.2-4] for a solution of a linear differential equation using $\|p^n\|^{-2} {p^n}^\top (A + A^\top) p^n = -\,\|p^n\|^{-2} d\,\|p^n\|^2/dt$. ■

Thus the system, which describes parameters $P(\cdot)$ and $\pi(\cdot)$ of the tubes $\mathcal{P}(\cdot) \in \mathfrak{P}^3$, is broken into n equations (13) for p^i, equation (18) for π_n (which can be integrated in the form (19)), and $n-1$ relations for the other values π_i (see (16), (17)).

Relations (13) ensure the following attractive properties of the tubes $\mathcal{P}(\cdot) \in \mathfrak{P}^3$.

Theorem 2 *The tubes $\mathcal{P}(\cdot) \in \mathfrak{P}^3$ are tight (in directions $p^n(t)$) outer estimates for $\mathcal{X}(t)$, $t \in \mathcal{T}$, and $\mathcal{X}(t) = \bigcap\{\mathcal{P}(t) \mid V \in \mathcal{V}\}$, $t \in \mathcal{T}$. Here, $\mathcal{V} \subset \mathbb{R}^{n \times n}$ is an arbitrary set of matrices V satisfying (13), where v^n range over all $v^n \in \mathbb{R}^n$ with $\|v^n\| = 1$.*

Proof. Let us prove that

$$\rho(\pm p^n(t)|\mathcal{P}(t)) = \rho(\pm p^n(t)|\mathcal{X}(t)), \quad t \in \mathcal{T}, \tag{20}$$

i.e., $\mathcal{P}(t)$ are tight (in directions $p^n(t)$) estimates for $\mathcal{X}(t)$. To simplify arguments, we put here and everywhere below in the arguments (only), without loss of generality, $p_0 = 0$, $r(t) \equiv 0$. It is not difficult to see, using (14) and (13), (18), (8), that

$$\rho(\pm p^n(t)|\mathcal{P}(t)) = \mathrm{Abs}\,(p^n(t)^\top P(t))\,\pi(t) = \|p^n(t)\|^2 \pi_n(t); \tag{21}$$

$$\frac{d}{dt}(\|p^n\|^2\pi_n) = \mathrm{Abs}\,(p^{n\top}R)\rho, \quad \|v^n\|^2\pi_n(0) = \mathrm{Abs}\,(v^{n\top}P_0)\pi_0. \tag{22}$$

Using (21), (22), and (5), we obtain (20). The relations $\mathcal{X}(t) = \bigcap\{\mathcal{P}(t) \mid V \in \mathcal{V}\}$ are the consequence of (20) and the convexity of $\mathcal{X}(t)$. ∎

For the future, it is convenient to associate the nonsingular matrix V from (13) with an arbitrary matrix $\bar{V}$ such that

$$\bar{V} = \{\bar{v}^i\} \in \mathbb{R}^{n\times n}; \quad \bar{v}^i = v^i, \quad i = 1,\ldots,n-1; \quad \det\bar{V} \neq 0, \tag{23}$$

and to associate the corresponding pair of matrices $P(t)$ and $Q(t)$ satisfying (13) and

$$P(t) = \{p^i(t)\}, \quad Q(t) = \{q^i(t)\} = (P(t)^{-1})^\top \tag{24}$$

with the pair of matrices $P^0(t)$ and $Q^0(t)$ satisfying the following relations

$$\dot{P}^0 = A(t)P^0,\ t \in \mathcal{T};\ P^0(0) = \bar{V};\ P^0(t) = \{p^{0,i}(t)\}; \quad Q^0(t) = \{q^{0,i}(t)\} = (P^0(t)^{-1})^\top. \tag{25}$$

Lemma 2 *Under conditions (13), (23)-(25), the columns of $Q(t)$ satisfy the relations*

$$\begin{aligned} &q^i(t) = q^{0,i}(t) - \frac{q^{0,i}(t)^\top q^{0,n}(t)}{\|q^{0,n}(t)\|^2}q^{0,n}(t) = q^{0,i} - \frac{q^{0,i\top}p^n}{\|p^n\|^2}p^n, \quad i = 1,\ldots,n-1;\\ &q^n(t) = \|p^n(t)\|^{-2}p^n(t) = \kappa(t)q^{0,n}(t), \qquad t \in \mathcal{T}, \end{aligned} \tag{26}$$

where $\kappa(t)$ is some scalar factor, $\kappa(t) \neq 0$ for any $t \in \mathcal{T}$.

Proof. We have the collinearity of vectors p^n, q^n, and $q^{0,n}$ because they are orthogonal to $n-1$ linearly independent vectors $p^i = p^{0,i}$, $i = 1,\ldots,n{-}1$. It is easy to verify, by direct calculations using (14), that vectors q^i from (26) satisfy $q^{i\top}p^j = \delta_{ij}$, $i,j = 1,\ldots,n$, where δ_{ij} is the Kronecker delta. ∎

The rest of the paper is devoted to investigating the properties of boundedness and unboundedness of tubes $\mathcal{P}(\cdot) \in \mathfrak{P}^3$ on $\mathcal{T} = [0,\infty)$ under Assumption 2, which is assumed everywhere below. First, we formulate several auxiliary criteria of boundedness / unboundedness of tubes $\mathcal{P}(\cdot) \in \mathfrak{P}^3$ in terms of functions $P(\cdot)$ and $Q(\cdot)$, or $P^0(\cdot)$ and $Q^0(\cdot)$. Then, on this base, we obtain more concrete criteria of boundedness / unboundedness in terms of the matrices V (or $\bar{V}$), T, J, and the sets $\mathcal{P}_0$, $\mathcal{R}(\cdot)$.

Lemma 3 *Under Assumption 2, a tube $\mathcal{P}(\cdot) \in \mathfrak{P}$ is bounded if the following n functions*

$$s_i(t) = \|p^i(t)\|\pi_i(t), \quad i = 1,\ldots,n, \tag{27}$$

are bounded, and $\mathcal{P}(\cdot){\in}\mathfrak{P}$ is unbounded if at least one of these functions $s_i(t)$, $i{\in}\{1,\ldots,n\}$, is unbounded. For tubes $\mathcal{P}(\cdot) \in \mathfrak{P}^3$, the last function $s_n(t)$ turns out to be bounded; if, in addition, the sets $\mathcal{R}(t)$ are singletons ($\rho(t) \equiv 0$), then $s_n(t) \to 0$ as $t \to \infty$.

Proof. It is known [10] that the tube $\mathcal{P}(\cdot) \in \mathfrak{P}$ is bounded (unbounded) on $\mathcal{T} = [0,\infty)$ if and only if the vector function $q(t) = (\mathrm{Abs}\,P(t))\pi(t)$ is bounded (unbounded, respectively). We have

$$q_j(t) = \sum_{k=1}^n |p^k_j|\,\pi_k \le \sum_{k=1}^n \|p^k\|\,\pi_k = \sum_{k=1}^n s_k(t), \quad j = 1,\ldots,n. \tag{28}$$

Therefore, if all $s_k(t)$ are bounded, then $q(t)$ is bounded too. Conversely, if there exists some unbounded function s_k, then at least one of the functions $|p_j^k|\,\pi_k$, $j \in \{1,\ldots,n\}$, is unbounded, and from the first equality in (28), we see that the corresponding component $q_j(t)$ is unbounded.

For tubes $\mathcal{P}(\cdot) \in \mathfrak{P}^3$, we have $s_n(t) = \rho(\pm p^n(t)\,\|p^n(t)\|^{-1}\,|\,\mathcal{X}(t))$ (see (20) and (21)). Hence, we can obtain boundedness of $s_n(t)$ and the relation $s_n(t) \to 0$ as $t \to \infty$ (if $\mathcal{R}(t)$ are singletons) from the corresponding properties of the sets $\mathcal{X}(t)$ under Assumption 2. The same conclusions (with more accurate estimates) as well as several estimates, which will be used below, can be obtained by the following way.

Using (13), (9), (10), and the well known property of an exponential of similar matrices $\exp(SXS^{-1}) = S(\exp X)S^{-1}$ [2, p. 55], we have

$$C_1\|e^{-J^\top t}u^n\|^2 \le \|p^n\|^2 = {u^n}^\top e^{-Jt}TT^\top e^{-J^\top t}u^n \le C_2\|e^{-J^\top t}u^n\|^2,\ u^n = (T^\top)^{-1}v^n; \quad (29)$$

$$\begin{gathered} C_3\|e^{Jt}u^i\|^2 \le \|p^i\|^2 = {u^i}^\top e^{J^\top t}(T^{-1})^\top T^{-1}e^{Jt}u^i \le C_4\|e^{Jt}u^i\|^2, \\ u^i = Tv^i, \quad i = 1,\ldots,n{-}1, \end{gathered} \quad (30)$$

where C_1 and C_2 are minimal and maximal eigenvalues of the symmetric matrix $TT^\top$, and C_3 and C_4 are similar positive constants for the matrix $(T^{-1})^\top T^{-1}$.

From (19) and boundedness of $\mathcal{R}(\cdot)$, we have

$$s_n(t) = s_{n,1}(t) + s_{n,2}(t), \quad s_{n,1} = \frac{\|p^n(0)\|^2\,\pi_n(0)}{\|p^n(t)\|}, \quad s_{n,2} \le \mathrm{Const}\,\frac{1}{\|p^n(t)\|}\int_0^t \|p^n(\tau)\|\,d\tau, \quad (31)$$

where $s_{n,2}(t) \equiv 0$ if $\rho(t) \equiv 0$.

Let us concretize estimates (29) for $\|p^n(t)\|$ using (10). Namely, let us represent the vector u^n in the form $u^n = ((u_1^n)^\top,\ldots,(u_m^n)^\top)^\top$, where vectors $u_k^n = ((u_{k,1}^n)^\top,\ldots,(u_{k,\gamma_k}^n)^\top)^\top \in \mathbb{R}^{\nu_k\gamma_k}$, $k = 1,\ldots,m$, and $u_{k,j}^n \in \mathbb{R}^{\nu_k}$ are either scalars or two-dimensional vectors. Then

$$\|p^n(t)\|^2 = \sum_{k=1}^{m}{}' e^{-2\alpha_k t}\sum_{i=1}^{\gamma_k}\left\|\sum_{j=1}^{i}\sigma_j^i(-t)^{i-j}H_k(-t)^\top u_{k,j}^n\right\|^2, \quad (32)$$

where $H_k(t)$ is $H(t)$ from (10) at $\beta = \beta_k$. The prime in the sum over k means that this sum contains only those summands for which $\|u_k^n\| \ne 0$. To obtain an estimate from below, it is sufficient to keep and estimate in the sum over i in (32) only those summands, which correspond to $i = \gamma_k$. Let j_k be minimal of $j \in \{1,\ldots,\gamma_k\}$ for which $u_{k,j}^n \ne 0$. Then, using $H(t)H(t)^\top \equiv I$ and properties of polynomials, we can see that there exist a constant $C_5 > 0$ and $t_* > 0$ such that

$$\|p^n(t)\|^2 \ge C_5\sum_{k=1}^{m}{}' e^{-2\alpha_k t}t^{2(\gamma_k - j_k)}\|u_{k,j_k}^n\|^2, \quad \text{for } t > t_*. \quad (33)$$

It is also not difficult to see that there exists a constant $C_6 > 0$ such that

$$\|p^n(t)\|^2 \le C_6\sum_{k=1}^{m}{}' e^{-2\alpha_k t}t^{2(\gamma_k - j_k)}\|u^n\|^2, \quad t > 0, \quad (34)$$

where the values C_5, C_6, t_*, j_k depend on u^n, i.e. on V. Relations (33), (31), and $v^n \ne 0$ give $\|p^n(t)\| \to \infty$, $s_{n,1}(t) \to 0$ as $t \to \infty$. Inequalities (33) and (34) give an estimate from above for $s_{n,2}(t)$ of the form $s_{n,2}(t) \le \mathrm{Const}\,\psi(t)$, where $\psi(t)$ has an ∞/∞ indeterminacy as $t \to \infty$. Using l'Hôpital's rule, we obtain $\lim_{t\to\infty}\psi(t) \le \mathrm{Const}$. Hence, boundedness of $s_n(t)$ is proved. ■

Lemma 4 *A tube $\mathcal{P}(\cdot) \in \mathfrak{P}^3$ is bounded if the following $n-1$ functions*

$$a_i(t) = \|p^i(t)\| \int_0^t \|q^i(\tau)\| d\tau, \quad i = 1, \dots, n-1, \tag{35}$$

are bounded, where $p^i(t)$, $q^i(t)$ are columns of matrices $P(t)$, $Q(t)$ from (13), (24).

Proof. Using (27), (16), (17), and boundedness of $s_n(t)$, we have

$$s_i(t) = s_{i,1}(t) + s_{i,2}(t) + s_{i,3}(t), \quad s_{i,1}(t) = \|p^i(t)\| \, \pi_i(0), \quad s_{i,3}(t) \le \mathrm{Const}\, a_i(t),$$
$$s_{i,2}(t) \le \|p^i(t)\| \int_0^t \|q^i(\tau)\| \,\mathrm{Const}\, \|p^n(\tau)\| \, \pi_n(\tau) \, d\tau \le \mathrm{Const}\, a_i(t), \quad i = 1, \dots, n-1, \tag{36}$$

where $s_{i,3}(t) \equiv 0$ if $\rho(t) \equiv 0$. It remains to take into account that $\|p^i(t)\| \to 0$ as $t \to \infty$ (due to (13), (7)) and to apply Lemma 3. ■

Lemma 5 *If $n = 2$, $\mathcal{P}(\cdot) \in \mathfrak{P}^3$ is bounded if the two following functions are bounded:*

$$c_1(t) = \|p^1(t)\| \int_0^t \|p^1(\tau)\|^{-1} \, d\tau, \quad c_2(t) = \|p^2(t)\|^{-1} \int_0^t \|p^2(\tau)\| \, d\tau. \tag{37}$$

Proof. If $n = 2$, matrices $P(t)$ are orthogonal, and $q^i(t) = p^i \, \|p^i\|^{-2}$, $i = 1, 2$. Hence, $a_1(t) = c_1(t)$ and boundedness of $c_1(t)$ is sufficient for boundedness of $\mathcal{P}(\cdot) \in \mathfrak{P}^3$ due to Lemma 4. Boundedness of $c_2(t)$ ensures boundedness of $s_2(t)$ due to (31) and the relation $\|p^2(t)\| \to \infty$ (boundedness of $s_2(t)$ has already been proved in Lemma 3). ■

Theorem 3 *If $n = 2$, then all tubes $\mathcal{P}(\cdot) \in \mathfrak{P}^3$ are bounded. If, in addition, the sets $\mathcal{R}(t)$ are singletons ($\mathcal{R}(t) \equiv r(t)$), then estimates $\mathcal{P}(t)$ from $\mathfrak{P}^3$ contract to their centers $p(t)$ as $t \to \infty$ for any $P(0) = V$.*

Proof. From (30), we have $c_1(t) \le \mathrm{Const} \int_0^t \varphi(\tau) \, d\tau \, / \, \varphi(t)$, where $\varphi(t) = ({u^1}^\top e^{J^\top t} \cdot e^{Jt} u^1)^{-1/2}$. Using (9) and (10), the function $\varphi(t)$ can be written in the following form: $\varphi(t) = (\sum_{k=1}^2 e^{2\lambda_k t} (u_k^1)^2)^{-1/2}$ in case A; $\varphi(t) = e^{-\alpha t} ({u^1}^\top F(t) u^1)^{-1/2}$, where $F(t) = \begin{bmatrix} 1 & t \\ t & t^2{+}1 \end{bmatrix}$ in case B; $\varphi(t) = e^{-\alpha t} \|u^1\|^{-1}$ in case C. Using l'Hôpital's rule in each of these cases, we obtain boundedness of $\mathcal{P}(\cdot)$.

Note that $s_i(t)$ are values of semi-axes of parallelepipeds $\mathcal{P}(t)$ with the normalized directions of their semi-axes $p^i(t)/\|p^i(t)\|$. Let $\rho(t) \equiv 0$. Taking into account (36), (27), (31), the convergence $\|p^1(t)\| \to 0$, and Lemma 3, it is sufficient to verify that $s_{1,2}(t) \to 0$ as $t \to \infty$, and we have $s_{1,2}(t) \le \mathrm{Const}\, \|p^1(t)\| \int_0^t \|p^1(\tau)\|^{-1} \|p^2(\tau)\|^{-1} \, d\tau$. Also, we have $C_7 \varphi(t) \le \|p^1(t)\|^{-1} \le C_8 \varphi(t)$, $\|p^2(t)\|^{-1} \le C_9 \psi(t)$, where $\varphi(t)$ is described above, $\psi(t) = (\sum_{k=1}^2 e^{-2\lambda_k t} (u_k^2)^2)^{-1/2}$ in case A; $\psi(t) = e^{\alpha t} ({u^2}^\top \Psi(t) u^2)^{-1/2}$, $\Psi(t) = \begin{bmatrix} t^2{+}1 & -t \\ -t & 1 \end{bmatrix}$ in case B; $\psi(t) = e^{\alpha t} \|u^2\|^{-1}$ in case C. Using $u^1 \ne 0$, $u^2 \ne 0$ (and, for case B, the relations $y^\top F(t) y \ge (y_1)^2$ if $y_2 = 0$, $y^\top F(t) y \ge (y_2)^2$ otherwise, which hold for any $y = (y_1, y_2)^\top$, and the similar ones for $\Psi(t)$), we can obtain, by direct calculations, the estimates of the type $s_{1,2}(t) \le \mathrm{Const}\, t^\zeta e^{-\mathrm{m} t}$, where $\zeta = 1$ for cases A and C, and $\zeta = 2$ for case B. ■

This result is unlike to the properties of two other subfamilies $\mathfrak{P}^i$ considered above, because there are two-dimensional systems for which all the estimates from $\mathfrak{P}^1$ and $\mathfrak{P}^2$ are unbounded (these systems are of different kinds for $\mathfrak{P}^1$ and $\mathfrak{P}^2$ — see the black squares in Figure 1).

Now, let us return to the general case $n \geq 2$.

Lemma 6 *A tube $\mathcal{P}(\cdot) \in \mathfrak{P}^3$ is bounded (unbounded) if the vector function $\tilde{q}(t) = (\mathrm{Abs}\,\tilde{P}(t))\,\tilde{\pi}(t)$ is bounded (unbounded), where $\tilde{P}(t) = \{p^1, \ldots, p^{n-1}, 0\} \in \mathbb{R}^{n\times n}$, the vectors $p^i(t)$, $i = 1, \ldots, n-1$, satisfy (13), and $\tilde{\pi}(t) \in \mathbb{R}^n$ satisfies (17) and (18).*

Proof. This follows from (23), (25), the relation $q(t) = \tilde{q}(t) + (\mathrm{Abs}\,p^n(t))\,\pi_n(t)$, and Lemma 3. ■

Corollary 1 *Let $P^0(t)$ satisfy (23) and (25). A tube $\mathcal{P}(\cdot) \in \mathfrak{P}^3$ is bounded if the vector function $q^0(t)=$ $(\mathrm{Abs}\,P^0(t))\,\tilde{\pi}(t)$ is bounded.*

Proof. It is evident because $\tilde{q}(t) \leq q^0(t)$. ■

Lemma 7 *A tube $\mathcal{P}(\cdot) \in \mathfrak{P}^3$ is bounded if the following $n-1$ functions*

$$b_i(t) = \|p^{0,i}(t)\| \int_0^t \|q^{0,i}(\tau)\|\,d\tau, \quad i = 1, \ldots, n-1, \tag{38}$$

are bounded, where $p^{0,i}$ and $q^{0,i}$ are columns of matrices P^0 and Q^0 from (23), (25).

Proof. From (17) and boundedness of the function $s_n(t)$, we have $\tilde{\pi}(t) \leq \pi(0) + \mathrm{Const}\int_0^t \mathrm{Abs}\,P(\tau)^{-1}\,\mathrm{e}\,d\tau$. Then, due to (7) and Corollary 1, it is sufficient to verify the boundedness of the vector function $\hat{q}(t) = \mathrm{Abs}\,P^0(t)\int_0^t \mathrm{Abs}\,P(\tau)^{-1}\,\mathrm{e}\,d\tau$. Using Lemma 2, we have

$$\begin{aligned} \hat{q}_i(t) &= \sum_{k=1}^{n} |p_i^{0,k}| \int_0^t \sum_{j=1}^{n} |q_j^k|\,d\tau \leq \mathrm{Const} \sum_{k=1}^{n} \|p^{0,k}\| \int_0^t \|q^k\|\,d\tau \\ &\leq \mathrm{Const}\Big(\sum_{k=1}^{n-1} 2\,b_k(t) + \|p^{0,n}(t)\| \int_0^t \frac{d\tau}{\|p^n(\tau)\|}\Big), \quad i = 1, \ldots, n, \end{aligned} \tag{39}$$

where $\|p^{0,n}(t)\| \int_0^t \|p^n(\tau)\|^{-1}\,d\tau \leq \mathrm{Const}\,e^{(-\mathrm{m}+\varepsilon)t} \to 0$ as $t \to \infty$ due to (7) and (33). ■

Proposition 3 *If $\mathcal{P}(\cdot) \in \mathfrak{P}^3$, then $\mu(\mathcal{P}(t)) \leq \mathrm{Const}\,e^{(\mathrm{M}-\mathrm{m}+\varepsilon)t}$, $t \geq 0$, for any $\varepsilon > 0$, where* Const > 0 *is some constant depending on ε, V, $\mathcal{P}_0$, and $\mathcal{R}(\cdot)$. Hence, the exponent $\chi(\mathcal{P})$ satisfies the inequality $\chi(\mathcal{P}) \leq \mathrm{M} - \mathrm{m}$ (similarly for tubes from $\mathfrak{P}^1$).*

Proof. Using estimates from the proofs of Lemmas 6, 7, 3, and Corollary 1, we have the following rough estimate for $q(t)$: $\|q(t)\| \leq \mathrm{Const}\,\|P^0(t)\| \int_0^t \|Q^0(\tau)\|\,d\tau$, which, together with (7), ensure the mentioned estimate for the tube exponent. ■

Two following lemmas give some sufficient conditions for unboundedness of tubes $\mathcal{P}(\cdot) \in \mathfrak{P}^3$.

Lemma 8 *Under the nondegeneracy condition (11), a tube $\mathcal{P}(\cdot) \in \mathfrak{P}^3$ is unbounded if at least one of $n-1$ functions $a_i(t)$ from (35) is unbounded.*

Proof. Due to Lemma 6 and (17), it is sufficient to obtain unboundedness of the vector-function $\mathrm{Abs}\, \tilde{P}(t) \int_0^t \mathrm{Abs}\, (P(\tau)^{-1} R(\tau)) \rho(\tau)\, d\tau$, but this follows from relations $\tilde{p}^i(t) \equiv p^i(t)$, $|q^i(t)^\top R(t)\rho(t)| = \rho(q^i(t)) \,|\, \mathcal{R}(t) - r(t)) \geq \rho(q^i(t)) \,|\, \mathcal{P}(0, I, \varepsilon_0 \mathrm{e})) \geq \mathrm{Const}\ \varepsilon_0 \, \|q^i(t)\|$, $i = 1, \ldots, n{-}1$, and conditions of Lemma 8. ■

Lemma 9 *Under the nondegeneracy condition (11), a tube $\mathcal{P}(\cdot) \in \mathfrak{P}^3$ is unbounded if at least for one of $i \in \{1, \ldots, n-1\}$, the function $b_i(t)$ from (38) is unbounded, and in addition, the following inequality is valid for the same i, some $\varepsilon > 0$, and for all sufficiently large $t > 0$:*

$$d_i(t) = 1 - \frac{(q^{0,i}(t)^\top q^{0,n}(t))^2}{\|q^{0,i}(t)\|^2 \|q^{0,n}(t)\|^2} = 1 - \frac{(q^{0,i}(t)^\top p^n(t))^2}{\|q^{0,i}(t)\|^2 \|p^n(t)\|^2} \geq \varepsilon. \tag{40}$$

Proof. This follows from equalities $\|q^i(t)\|^2 = \|q^{0,i}(t)\|^2\, d_i(t)$ (which are true due to Lemma 2) and Lemma 8. ■

Now let us formulate boundedness conditions in terms of matrices V. Although it is possible to mark out some special cases when estimates $\mathcal{P}(\cdot) \in \mathfrak{P}^3$ turn out to be bounded for the systems with defective matrices (see, for example, Theorem 3), it is seemingly useless to hope to obtain a similar good result in the general case of systems with defective matrices because $n-1$ columns of the orientation matrices satisfy the same relations as for the tubes $\mathcal{P}(\cdot) \in \mathfrak{P}^1$. Therefore, we concentrate our attention on systems with diagonalizable matrices. In this connection, the following lemma may be useful for investigating the case when the matrix A is diagonalizable. It also can be used for verifying the conditions of Theorem 1.

Lemma 10 *Let $\tilde{V} \in \mathbb{R}^{n \times n}$ be an arbitrary nonsingular matrix and $\tilde{W} = \tilde{V}^{-1}$. Let the set of indices $\{1, \ldots, n\}$ be partitioned into m consecutive groups such that each group consists of ν_k elements, where $\nu_k = 1$ or $\nu_k = 2$, $k = 1, \ldots, m$. Let us consider two ways to represent the matrices $\tilde{V}$ and $\tilde{W}$ in block-wise form:*

(1) $\tilde{V}$ and $\tilde{W}$ are decomposed into the corresponding blocks $\tilde{V}_i^j \in \mathbb{R}^{\nu_i \times \nu_j}$, $\tilde{W}_j^i \in \mathbb{R}^{\nu_j \times \nu_i}$, $i, j = 1, \ldots, m$;

(2) $\tilde{V}$ and $\tilde{W}$ are represented by their columns and rows correspondingly: $\tilde{V} = \{\tilde{v}^l\}$, $\tilde{W}^\top = \tilde{S} = \{\tilde{s}^l\}$, where vectors $\tilde{v}^l, \tilde{s}^l \in \mathbb{R}^n$, in turn, are decomposed into vectors of dimensions ν_k: $\tilde{v}^l = ((\tilde{v}_1^l)^\top, \ldots, (\tilde{v}_m^l)^\top)^\top$, $\tilde{s}^l = ((\tilde{s}_1^l)^\top, \ldots, (\tilde{s}_m^l)^\top)^\top$, where $\tilde{v}_k^l, \tilde{s}_k^l \in \mathbb{R}^{\nu_k}$.

Then, for any $i, j \in \{1, \ldots, m\}$, relations (41) and (42) are equivalent:

$$Z_i^j = \sum_{k=1}^{m} \mathrm{Abs}\, \tilde{V}_i^k \, \mathrm{Abs}\, \tilde{W}_k^j = 0 \in \mathbb{R}^{\nu_i \times \nu_j}; \tag{41}$$

$$\|\tilde{v}_i^l\| \cdot \|\tilde{s}_j^l\| = 0, \quad l = 1, \ldots, n. \tag{42}$$

Proof. Relation (41) is equivalent to m equalities $\mathrm{Abs}\, \tilde{V}_i^k \, \mathrm{Abs}\, \tilde{W}_k^j = 0 \in \mathbb{R}^{\nu_i \times \nu_j}$, $k = 1, \ldots, m$. We have $\tilde{V}_i^k = \tilde{v}_i^{l_k} \in \mathbb{R}^{\nu_i \times 1}$ for some $l_k \in \{1, \ldots, n\}$ if $\nu_k = 1$, and $\tilde{V}_i^k = \{\tilde{v}_i^{l_k}, \tilde{v}_i^{l_k+1}\} \in \mathbb{R}^{\nu_i \times 2}$ for some $l_k \in \{1, \ldots, n{-}1\}$ if $\nu_k = 2$. Similarly, $\tilde{W}_k^j = (\tilde{s}_j^{l_k})^\top \in \mathbb{R}^{1 \times \nu_j}$ if $\nu_k = 1$, and $\tilde{W}_k^j = \{\tilde{s}_j^{l_k}, \tilde{s}_j^{l_k+1}\}^\top \in \mathbb{R}^{2 \times \nu_j}$ if $\nu_k = 2$. If k ranges over values $\{1, \ldots, m\}$, then the index l of vectors $\tilde{v}_i^l$, $\tilde{s}_j^l$ in (42) ranges over values $\{1, \ldots, n\}$ and vice versa. Therefore, the lemma can be obtained by considering eight possible forms of matrices $\mathrm{Abs}\, \tilde{V}_i^k \, \mathrm{Abs}\, \tilde{W}_k^j$ corresponding to different cases $\nu_i \in \{1, 2\}$, $\nu_j \in \{1, 2\}$, $\nu_k \in \{1, 2\}$ and by direct calculations, which we omit here. ■

Theorem 4 *Let the matrix A be diagonalizable and let $\mathcal{P}(\cdot) \in \mathfrak{P}^3$ be a tube corresponding to the matrix $P(0) = V$ from (13). Then the following situations are possible.*

(1) If $\mathrm{M} = \mathrm{m}$, then $\mathcal{P}(\cdot)$ is bounded for any V.

(2) Let $\mathrm{M} \neq \mathrm{m}$, T be the matrix from (9), and $\bar{V}$ be an arbitrary matrix satisfying (23). Let matrices $\tilde{V} = T\bar{V}$ and $\tilde{W} = \tilde{V}^{-1} = \bar{V}^{-1}T^{-1}$ be decomposed into blocks as described in Lemma 10 (the enumeration of blocks and eigenvalues λ_i below is determined by the matrix T and (9)). If V is such that, for any pair of eigenvalues λ_i and λ_j satisfying the inequality $|\mathrm{Re}\,\lambda_i| < |\mathrm{Re}\,\lambda_j|$, the following equalities are valid:

$$\|\tilde{v}_i^l\| \cdot \|\tilde{s}_j^l\| = 0, \quad l = 1, \ldots, n-1, \tag{43}$$

(or any of more strong conditions (42) or (41) are valid), then the tube $\mathcal{P}(\cdot)$ is bounded.

Proof. Let us verify the conditions of Lemma 7. Similarly to (30), we have

$$\|p^{0,i}\| \leq \mathrm{Const}\, \|e^{Jt}\tilde{v}^i\|, \quad \|q^{0,i}\| \leq \mathrm{Const}\, \|e^{-J^\top t}\tilde{s}^i\|, \quad i = 1, \ldots, n. \tag{44}$$

Using the structure of exponentials of diagonalizable matrices and equivalence of vector norms, we obtain

$$b_i(t) \leq \mathrm{Const} \sum_{k=1}^{m} e^{\alpha_k t}\|\tilde{v}_k^i\| \int_0^t \sum_{l=1}^{m} e^{-\alpha_l \tau}\|\tilde{s}_l^i\|\, d\tau, \quad i = 1, \ldots, n.$$

Under $\mathrm{m} = \mathrm{M}$, boundedness of all $b_i(t)$ is evident; otherwise it follows, from the assumptions of the theorem, from the inequalities

$$\lim_{t\to\infty} b_i(t) \leq \mathrm{Const} \sum_{k,l=1}^{m} e^{(\alpha_k - \alpha_l)t}\|\tilde{v}_k^i\|\, \|\tilde{s}_l^i\|, \quad i = 1, \ldots, n,$$

which can be obtained by l'Hôpital's rule. ■

Corollary 2 *If $\bar{V} = T^{-1}$, then $Z_i^j{=}0$ for any i, j, $i{\neq}j$, and the above sufficient conditions for boundedness are satisfied. Therefore, it is clear how to find at least $C_{n-1}^n = n$ matrices $P(0) = V$ such that the corresponding tubes $\mathcal{P}(\cdot) \in \mathfrak{P}^3$ are bounded: we can construct such V if we take $n{-}1$ columns of T^{-1} as the first $n{-}1$ columns of V (then the last column of V is determined uniquely, up to a scalar factor, because it is orthogonal to the previous ones). We have n choices for the first $n{-}1$ columns of T^{-1}.*

Lemma 9 provides the following simple proposition.

Proposition 4 *Let $n \geq 3$, the matrix A be diagonalizable, all its eigenvalues be real, and $\mathrm{m} \neq \mathrm{M}$. Then, under the nondegeneracy condition (11), there exist matrices V that generate unbounded tubes $\mathcal{P}(\cdot) \in \mathfrak{P}^3$.*

Proof. Let us consider a matrix V, which is associated (according to (13), (23)) with the matrix $\bar{V} = T^{-1}\tilde{V}$, where $\tilde{V}$ has a specific structure similar to $\tilde{V}$ from [10, proof of Theorem 3]. In (9), we have, without loss of generality, $|\lambda_2| > |\lambda_1|$. Let $\tilde{V} = \{\tilde{v}_i^j\}$ $(i, j = 1, \ldots, n)$ be such that all diagonal elements of $\tilde{V}$ and also the element $\tilde{v}_1^{i_*}$ are equal to 1, where $i_* = \nu_1 + 1 = 2$, and the other elements are zeros. Let us attempt to apply Lemma 9 with $i = i_* = 2$ assuming that $n > i_*$. It is not difficult to verify that

$\tilde{v}^{i_*} = (1, 1, 0, \dots, 0)^\top$, $\tilde{s}^{i_*} = \mathrm{e}^{i_*}$, $\tilde{s}^n = \mathrm{e}^n$. Then, similarly to the proof of Theorem 4, we have

$$\|p^{0,i_*}(t)\|^2 \geq \mathrm{Const}\,(e^{2\lambda_1 t} + e^{2\lambda_2 t}), \quad \|q^{0,i_*}(t)\|^2 \geq \mathrm{Const}\; e^{-2\lambda_2 t},$$

and, consequently, unboundedness of $b_{i_*}(t)$. Also, it is not difficult to verify that

$$d_{i_*}(t) \equiv d_{i_*} = 1 - \frac{((TT^\top)^n_{i_*})^2}{(TT^\top)^{i_*}_{i_*}(TT^\top)^n_n} = 1 - \frac{((g^{i_*})^\top g^n)^2}{\|g^{i_*}\|^2\|g^n\|^2},$$

where we use the notation $T^\top = G = \{g^i\}$. Evidently, $d_{i_*} > 0$ (because the equality $d_{i_*} = 0$ is possible only if g^{i_*} and g^n are collinear, but they are the columns of the nonsingular matrix $T^\top$), and we have (40) for $i = i_*$. We can apply Lemma 9. ■

Let us formulate some conditions, which ensure contracting the sets $\mathcal{P}(t)$ to their centers.

Let matrices $V = \{v^i\}$, $\tilde{V} = \{\tilde{v}^i\}$, and $\tilde{S} = \{\tilde{s}^i\} = \tilde{W}^\top$ be as in Theorem 4, and the columns of $\tilde{V}$ and $\tilde{S}$ be decomposed as is described in Lemma 10. Let the vector $u^n = (T^\top)^{-1}v^n \in \mathbb{R}^n$ be decomposed similarly to vectors $\tilde{v}^l$ and $\tilde{s}^l$:

$$u^n = (T^\top)^{-1}v^n = ((u^n_1)^\top, \dots, (u^n_m)^\top)^\top,$$

where vectors $u^n_k \in \mathbb{R}^{\nu_k}$, $k = 1, \dots, m$. Let us denote

$$\delta^* = \max\{|\mathrm{Re}\,\lambda_j| \mid j \in \{1, \dots, m\},\ \|u^n_j\| \neq 0\},$$

$$\delta_i = \max\{|\mathrm{Re}\,\lambda_l| - |\mathrm{Re}\,\lambda_k| \mid k, l \in \{1, \dots, m\},\ \|\tilde{v}^i_k\| \cdot \|\tilde{s}^i_l\| \neq 0\}, \quad i = 1, \dots, n-1.$$

Proposition 5 *Let the matrix A be diagonalizable and the sets $\mathcal{R}(t)$ be singletons. If V is such that*

$$-\delta^* + \delta_i < 0, \quad i = 1, \dots, n-1, \tag{45}$$

then the sets $\mathcal{P}(t)$ contract to their centers $p(t)$ as $t \to \infty$.

Proof. Due to Lemma 3 and (36), it is sufficient to verify that $s_{i,2}(t) \to 0$ as $t \to \infty$, $i = 1, \dots, n-1$. Using relations $\|q^i(t)\| \leq 2\|q^{0,i}(t)\|$ for $i = 1, \dots, n-1$ (see Lemma 2), (44), (31), (29), we can obtain, similarly to the proof of Theorem 4, that

$$s_{i,2}(t) \leq \mathrm{Const} \sum_{k=1}^m e^{\alpha_k t}\|\tilde{v}^i_k\| \int_0^t \frac{\sum_{l=1}^m e^{-\alpha_l \tau}\|\tilde{s}^i_l\|}{\sum_{j=1}^m e^{-\alpha_j \tau}\|\tilde{u}^n_j\|}\,d\tau, \quad i = 1, \dots, n-1. \tag{46}$$

If, for some $i \in \{1, \dots, n-1\}$ the integral in (46) is bounded, then the relation $s_{i,2}(t) \to 0$ is evident for this i. Otherwise, by l'Hôpital's rule, we have

$$\lim_{t\to\infty} s_{i,2}(t) \leq \mathrm{Const} \sum_{k,l=1}^m e^{(\alpha_k - \alpha_l)t}\|\tilde{v}^i_k\|\,\|\tilde{s}^i_l\|\Big(\sum_{j=1}^m e^{-\alpha_j t}\|\tilde{u}^n_j\|\Big)^{-1}, \tag{47}$$

and (45) ensures that the right-hand side of (47) tends to 0 as $t \to \infty$. ■

Corollary 3 *Under conditions of Theorem 4, if the sets $\mathcal{R}(t)$ are singletons, then the sets $\mathcal{P}(t)$ contract to their centers $p(t)$ as $t \to \infty$.*

Proof. Under the conditions of Theorem 4, the numerators in (47) are bounded for $i = 1, \dots, n-1$; therefore $s_{i,2}(t) \to 0$ as $t \to \infty$, $i = 1, \dots, n-1$. ■

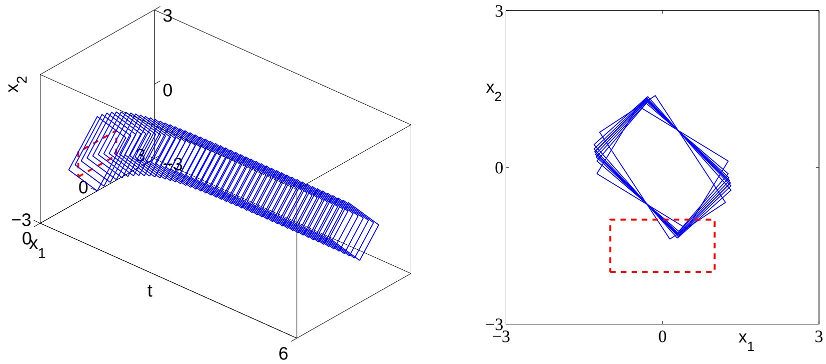

Figure 2: Estimates in Example 2

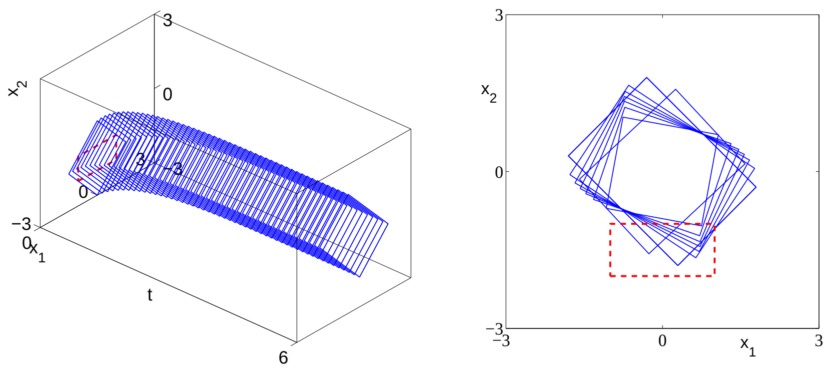

Figure 3: Estimates in Example 3

6 Examples of Estimates from $\mathfrak{P}^3$

In [11], examples of constructing estimates from $\mathfrak{P}^1$ and $\mathfrak{P}^2$ for two-dimensional systems of five types were presented with corresponding illustrations that also demonstrate the form and size of reachable sets in the examples. Here, we consider the same systems and present the estimates from $\mathfrak{P}^3$ in Figures 2–5 to have an occasion for comparison with figures from [11]. The estimates are calculated using the Euler approximations similarly to [8], with $N = 300$ time steps in almost all examples.

In the left-hand side of each figure, we demonstrate some tube $\mathcal{P}(\cdot) \in \mathfrak{P}^3$ calculated for the corresponding example. The right-hand side of each figure shows the initial set (dashed line) and several tight outer estimates for the reachable set at time $t = \theta$. The estimates correspond to several orthogonal matrices V of the form $V = \{v^1, v^2\}$, $v^1 = (\cos\varphi, \sin\varphi)^\top$, $v^2 = (-\sin\varphi, \cos\varphi)^\top$, where φ runs through a uniform grid of angles.

In Examples 1–3, we set $\mathcal{P}_0 = \mathcal{P}((0, -1.5)^\top, I, (1, 0.5)^\top)$, $\mathcal{R}(t) \equiv \mathcal{P}(0, I, (0.5, 1)^\top)$. We set $\theta = 6$ in all examples.

Example 1 $A = -I$. We have case A with $\lambda_1 = \lambda_2 = -1$. Estimates from the

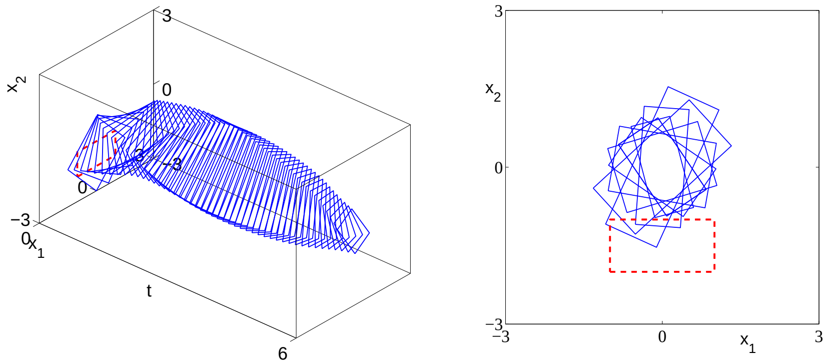

Figure 4: Estimates in Example 4

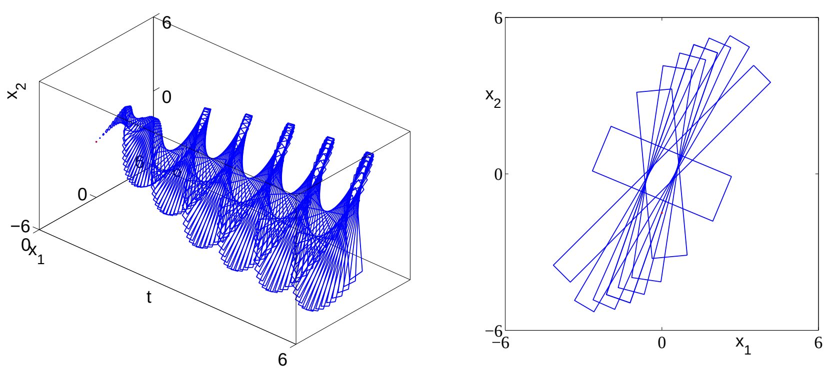

Figure 5: Estimates in Example 5

three families $\mathfrak{P}^i$, $i = 1, 2, 3$, look the same in this example (see [11, Figure 1]).

Example 2 $A = \begin{bmatrix} -1.2 & -0.2 \\ -0.3 & -1.3 \end{bmatrix}$. We have case A with $\lambda_1 = -1$, $\lambda_2 = -1.5$.

Example 3 $A \equiv \begin{bmatrix} -0.8 & 0.2 \\ -0.2 & -1.2 \end{bmatrix}$. We have case B with $\lambda_1 = \lambda_2 = -1$. Note that there is a misprint in the data of [11, Example 3]. In fact, the data for [11, Figure 5] are the same as in the present Example 3.

Example 4 $A \equiv \begin{bmatrix} -0.5 & -0.5 \\ 1 & -1.5 \end{bmatrix}$. We have case C with $\alpha = -1$, $\beta = 0.5$, and $|\beta| < |\alpha|$; $\mathcal{P}_0 = \mathcal{P}((0, -1.5)^\top, I, (1, 0.5)^\top)$, $\mathcal{R} \equiv \mathcal{P}(0, I, (0, 1)^\top)$.

Example 5 $A \equiv \begin{bmatrix} 2.5 & -3.5 \\ 7 & -4.5 \end{bmatrix}$. We have case C with $\alpha = -1$, $\beta = 3.5$, and $|\beta| > |\alpha|$; $\mathcal{P}_0 = \mathcal{P}((0, -1.5)^\top, I, 0)$ (it is a singleton), $\mathcal{R} \equiv \mathcal{P}(0, I, (0, 1)^\top)$; $N = 3600$. Note that although the depicted estimates correspond to the bounded tubes $\mathcal{P}(\cdot) \in \mathfrak{P}^3$, each of them provides quite a rough estimate of the reachable set at time $t = \theta$ in this example, but in aggregate, they outline the reachable set.

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