Computation of Potential and Attraction Force of an Ellipsoid*

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Abstract

The paper examines the problem of computing the potential and the attraction force of an ellipsoid, which requires taking a triple integral with an analytically integrable kernel. We consider the kernel as a weight function, while the inner integral is approximated by a quadrature for the product of functions, one of which has an integrable singularity. Such an approach makes it possible to obtain an integrand with only a weak logarithmic singularity before carrying out the next integration. This singularity can be overcome easily by changing a variable in the next outer integrals. Thus, to compute all the integrals, quadrature formulas without singularities are obtained. In addition, the functions to be computed do not have large values at the integration points. To carry out numerical experiments, complicated test functions are constructed. These functions are the exact potential and the exact force of attraction of an ellipsoid of rotation with an elliptic density distribution.

Keywords: ellipsoid, potential, attraction force, numerical quadrature
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1 Introduction

The problem of finding the volume potential and the attraction force is a classical problem of modern mathematical physics, and its first solutions for bodies with simple forms were obtained by Sir Isaac Newton. By the present time, precise solutions for bodies of various forms with layered non-uniform distribution of densities are known. A sufficiently complete account of such solutions can be found in [6]. For bodies having ellipsoidal forms with an arbitrary distribution of density, precise solutions are expressed through the Maclaurin series [8], which are hardly acceptable in numerical computation of the volume potential.

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The most common approach to the numerical computation of the potential is to approximate the density by a family of basic functions in such a way that the calculation of the linear integral operators obtained becomes possible in the sense that it does not require coping with singularities in integrands. Such basic functions may be spherical harmonics [9], piecewise-polynomial approximations [2] or exponential type functions depending on the distance between the nodes as well as the points where the potential is to be computed [7]. However, such approaches require a certain smoothness of the density in the whole space. In the case when the density is nonzero only inside the volume of a body, the above approaches to computation of the potential lead to essential errors near the boundary of the body. For the solution of the above problem, a bounded layer is introduced in [5], where an extra approximation is done. The cubature formulas for computation of a volume potential are presented, but they are not confirmed by any numerical experiments. Another approach presents the potential as an inverse Fourier transform from the product of the Fourier images of the singular kernel and density [4]. We have to note that, along with the problem of computation of the Fourier image from the singular kernel, one should take into account the fact that a discrete analog of the convolution theorem requires the periodicity of integrands.

In this paper, we present a simple, both in its concept and in its numerical implementation, semi-analytical method for the computation of the potential and the attraction force of a bodies having elliptical forms with density defined on a non-uniform grid. The main idea of the method is in representing required functions by the triple integrals by such a way that we can compute analytically the inner integrals from the kernels. In this approach, the kernel is considered to be a weight function. To approximate the inner integral, the quadrature formula for the product of functions, one of which has an integrable singularity, is proposed. This approach allows us to obtain an integrand with a weak logarithmic singularity when computing the second integral. This singularity can be overcome easily by changing variables in subsequent outer integrals. In computing one component of the attraction force, the inner integrand has a stronger singularity. In this case the quadrature for the product of functions with singularities is applied not only for the inner integral, but also for computing the integral with respect to the second variable.

To find all the integrals, we obtain quadrature formulas without singularities, and the functions to be calculated do not have large values at the quadrature nodes.

The method is illustrated by numerical experiments applied to sufficiently complicated test functions. These functions are the exact potential and the exact attraction force of an ellipsoid of rotation with an elliptic density distribution.

2 Approximately Computing the Integral for the Product of Functions

We consider the following quadrature for the computation of integrals:

$$\int_a^b f(x)\, g(x)\, dx \quad , \tag{1}$$

where $g(x) \in L^1[a,b]$ and $g(x) \geq 0$ when $x \in [a,b]$ and $f(x) \in C[a,b]$, provided that the integrals $G(\alpha,\beta) = \int_\alpha^\beta g(x)\, dx$ when α and $\beta \in [a,b]$ can be computed analytically.

To obtain the quadrature for the integral (1), we consider the function $g(x)$ as a weight function [1]. We define a grid with the nodes x_i, $i = 1, \ldots, n$, on the interval $[a, b]$, where the first and last nodes can coincide with the endpoints of the interval. We denote $x_{i+1/2} = \frac{x_i + x_{i+1}}{2}$, $h_i = x_{i+1} - x_i$, $i = 1, \ldots, n-1$.

We approximate the function $f(x)$ on every elementary sub-interval by the piecewise-constant function $L_0(x)$ as follows:

$$
\begin{aligned}
&L_0(x) = f(x_i) \quad \text{when} \quad x \in \left[x_{i-1/2}, x_{i+1/2}\right], \quad i = 2, \ldots, n-1,\\
&L_0(x) = f(x_1) \quad \text{when} \quad x \in \left[a, x_{3/2}\right),\\
&L_0(x) = f(x_n) \quad \text{when} \quad x \in \left(x_{n-1/2}, b\right].
\end{aligned}
$$

Then the approximation error on every elementary interval has the second order [1], and the total error $\varepsilon = \int\limits_a^b \left[f(x) - L_0(x)\right] g(x)\, dx$ of a combined rectangle formula with a weight function $g(x)$ satisfies the inequality

$$|\varepsilon| \le h_0 M_0 M_g ,$$

where $h_0 = \max\left\{x_1 - a,\ b - x_n,\ \frac{1}{2}\max\limits_i h_i\right\}$, $M_0 = \max\limits_{x \in [a,b]} |f'(x)|$, $M_g = G(a, b)$, and the constants M_0 and M_g do not depend on h_0.

When this approximating function for $f(x)$ is selected, the quadrature for computing the integral (1) has the form

$$
\begin{aligned}
&\int\limits_a^b f(x)\, g(x)\ dx \approx \int\limits_a^b L_0(x)\, g(x)\ dx\\
&\quad = f(x_1)\ G\left(a, x_{3/2}\right) + f(x_n)\, G\left(x_{n-1/2}, b\right) + \sum_{i=2}^{n-1} f(x_i)\, G\left(x_{i-1/2}, x_{i+1/2}\right)
\end{aligned}
\tag{2}
$$

3 Computing the Potential of an Ellipsoid

We consider a body T bounded by the elliptic surface, with the density ρ. We choose a system of coordinates with the origin in the center of the ellipsoid with axes that are oriented along the main axes of the ellipsoid. The potential of the body T at the point $M_0 = M_0(x_0, y_0, z_0)$ is defined by

$$U(M_0) = \int\limits_T \frac{\rho(M)}{|M - M_0|}\, d\tau , \tag{3}$$

where M is a current point of integration, and $d\tau$ is an element of the body volume. Let us pass to the spherical coordinates φ, θ, r. Then formula (3) takes the form

$$U(M_0) = \int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi} \sin\theta\, d\theta \int\limits_0^{R(\varphi,\theta)} \frac{r^2 \rho(\varphi, \theta, r)}{\sqrt{r_0^2 - 2 r r_0 \cos\psi + r^2}}\, dr , \tag{4}$$

where ψ is the angle between $\overrightarrow{OM}$ and $\overrightarrow{OM_0}$,

$$\cos\psi = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\varphi - \varphi_0).$$

Let the density ρ be a continuous function with respect to every variable. Then we can apply the quadrature that is described in the previous section for the numerical computation of the inner integral in (4). For fixed values of the variables $\varphi,\ \varphi_0,\ \theta,\ \theta_0$, we denote $f(r) = \rho(\varphi, \theta, r)$, $g(r) = \frac{r^2}{\sqrt{r_0^2 - 2rr_0\cos\psi + r^2}}$. Calculating the integrals from the function $g(r)$, we obtain

$$G(r_i, r_{i+1}) = \int\limits_{r_i}^{r_{i+1}} g(r)\, dr = G_0(r_{i+1}) - G_0(r_i)\ , \tag{5}$$

where

$$G_0(x) = \frac{1}{2}\left[(x + 3r_0\cos\psi)\, sq(x) + r_0^2\left(3\cos^2\psi - 1\right)\log(w(x))\right], \tag{6}$$

$$sq(x) = \sqrt{r_0^2 - 2r_0 x\cos\psi + x^2}\ , \quad w(x) = sq(x) + x - r_0\cos\psi. \tag{7}$$

To compute the complete inner integral in (4), we use the combined rectangle formula (2), where the integrals on elementary sub-intervals are calculated by the formula (5). Before proceeding to the numerical integration of the second integral in (4), we note that summands with singularities appear in the functions $G_0(r_i)$ after the first integration. However, it is just a weak logarithmic singularity that can be taken into account in the numerical integration by inserting a new variable. Indeed, let us consider a summand with singularity at the points $r_i = r_0$,

$$\mu(\varphi) = \log(w(r_0)) = \log\left(r_0\sqrt{1 - \cos\psi}\ \left[\sqrt{2} + \sqrt{1 - \cos\psi}\right]\right).$$

The argument of the logarithm vanishes when $\psi = 0$, i.e., when $\theta = \theta_0$ and $\varphi = \varphi_0$. If $\theta = \theta_0$,

$$\mu(\varphi) = \log\left(r_0\,|\sin\theta_0|\ \sqrt{1 - \cos(\varphi - \varphi_0)}\ \left[\sqrt{2} + |\sin\theta_0|\ \sqrt{1 - \cos(\varphi - \varphi_0)}\right]\right).$$

For the integration of the second integral in (4), we change the variable $\varphi - \varphi_0 = t^3$. It is easy to show that $\lim\limits_{t\to 0} t^2 \log\left(\sqrt{1 - \cos\ t^3}\right) = 0$, and the function $t^2 G_0(t, \theta, x)$ is continuous with respect to the new variable t. Then the numerical integration with respect to the variable t has no singularities.

Note that $\psi = 0$ for any values of φ if $\theta = \theta_0$ and $\theta_0 = 0$ or $\theta_0 = \pi$. Although the value of the first integral is multiplied by the factor $\sin\theta$ when we integrate over the second variable (see formula (4)), this singularity is removable. Because $\lim\limits_{\theta\to 0}\sin\theta\log(\alpha\sin\theta) = 0$, this singularity does not create any problems in the course of numerical integration.

4 Computing the Attraction Force of an Ellipsoid

The attraction force of a body in the spherical coordinate system is the vector

$$\vec{F} = \left(\frac{\partial U}{\partial\varphi_0}, \frac{\partial U}{\partial\theta_0}, \frac{\partial U}{\partial r_0}\right)^T.$$

In this section, we separately consider the computation of every component of the attraction force of the ellipsoid.

4.1 A Component of the Attraction Force along the Coordinate r_0

For fixed values of the variables φ, φ_0, θ, θ_0, we denote

$$f(r) = \rho(\varphi, \theta, r), \quad g(r) = \frac{\partial}{\partial r_0} \frac{r^2}{\sqrt{r_0^2 - 2rr_0 \cos\psi + r^2}} .$$

Then the function $G_0(x)$ in formula (5) has the form:

$$G_0(x) = \mu_r(x) + r_0 \left(3\cos^2\psi - 1\right) \log(w(x)), \tag{8}$$

where

$$\mu_r(x) = \frac{x^2 \cos\psi + xr_0 \left(1 - 6\cos^2\psi\right) + 3r_0^2 \cos\psi}{sq(x)} ,$$

and $sq(x)$ and $w(x)$ are defined in (7). The new singularity has appeared in formula (8), because the function $\mu_r(x)$ tends to infinity when $\psi = 0$ and $x = r_0$.

However, if a uniform grid is defined for the variable r, the first and the last nodes coincide with the boundary of the body, and the points where the component of the attraction force is computed are situated in the middle of the grid nodes. Then, for $\psi = 0$,

$$\mu_r(r_0 + \Delta r/2) - \mu_r(r_0 - \Delta r/2) = -6r_0,$$

and the functions $G(r_i, r_{i+1})$ in (5) have no singularities in addition to the logarithmic ones.

4.2 A Component of the Attraction Force along the Coordinate θ_0

For fixed values of the variables φ, φ_0, θ, θ_0, we denote

$$f(r) = \rho(\varphi, \theta, r), \quad g(r) = \frac{\partial}{\partial \theta_0} \frac{r^2}{\sqrt{r_0^2 - 2rr_0 \cos\psi + r^2}} .$$

Then the function $G_0(x)$ in formula (5) has the form:

$$G_0(x) = r_0 cos'\psi \left[\mu_\theta(x) + 3r_0 \cos\psi \log(w(x))\right], \tag{9}$$

where $cos'\psi = \frac{\partial \cos\psi}{\partial\theta_0}$, and

$$\mu_\theta(x) = \frac{x^2 - 6r_0 x \cos\psi + 3r_0^2}{sq(x)} + \frac{r_0 x \cos\psi - r_0^2 + r_0 sq(x)}{sq(x)\sin^2\psi} .$$

Comparing formula (9) with formulas (6) and (8), a new singularity has appeared in the second summand of the function $\mu_\theta(x)$. This singularity cannot be overcome during the next integration by changing variables or by choosing an appropriate grid. Therefore, to compute the second integral numerically, we apply the quadrature (2) again. In this case, when calculating the first integral, we multiply the function $G_0(x)$ by the factor $\frac{\sin^2\psi}{cos'\psi}$ and denote an approximate value of the first integral calculated with such a factor for the fixed values φ, φ_0, as $I_1(\theta, \theta_0)$. Then, introducing the notation

$$f(\theta) = I_1(\theta, \theta_0) \ , \quad g(\theta) = \frac{\sin\theta cos'\psi}{\sin^2\psi} ,$$

we can apply the quadrature (2) once again for computing the second integral. Having integrated the function $g(\theta)$ on every elementary interval $[\theta_j, \theta_{j+1}]$, we obtain

$$\int_{\theta_j}^{\theta_{j+1}} \frac{\sin\theta cos'\psi}{\sin^2\psi}\, d\theta = \frac{1}{\left(1-\sin^2\theta_0\sin^2\varphi_1\right)}\left[\mu_\theta^{(1)}(\theta_j,\theta_{j+1}) + \mu_\theta^{(2)}(\theta_j,\theta_{j+1}) + \mu_\theta^{(3)}(\theta_j,\theta_{j+1})\right] , \tag{10}$$

where $\varphi_1 = \varphi - \varphi_0$,

$$\mu_\theta^{(1)}(\theta_j,\theta_{j+1}) = -\sin(2\theta_0)\left|\sin\varphi_1\right|\cos\varphi_1\; v_1(\theta_j,\theta_{j+1}) \ ,$$

$$\mu_\theta^{(2)}(\theta_j,\theta_{j+1}) = 0.5\sin\theta_0\left[1-\sin^2\varphi_1\left(1+\cos^2\theta_0\right)\right]\; v_2(\theta_j,\theta_{j+1}) \ ,$$

$$\mu_\theta^{(3)}(\theta_j,\theta_{j+1}) = \cos\theta_0\cos\varphi_1\left(1+\sin^2\theta_0\sin^2\varphi_1\right)(\theta_{j+1}-\theta_j) \ ,$$

$$v_1(\theta_j,\theta_{j+1}) = \left[\arctan\frac{\left(1-\sin^2\theta_0\cos^2\varphi_1\right)\theta - 0.5\sin(2\theta_0)\cos\varphi_1}{\sin\theta_0\left|\sin\varphi_1\right|}\right]_{tg\theta_j}^{tg\theta_{j+1}} ,$$

$$v_2(\theta_j,\theta_{j+1}) = \left[\log\frac{\theta^2\left(1-\sin^2\theta_0\cos^2\varphi_1\right) - \theta\sin(2\theta_0)\cos\varphi_1 + \sin^2\theta_0}{1+\theta^2}\right]_{tg\theta_j}^{tg\theta_{j+1}} .$$

Integral (10) and the function $v_2(\theta_j,\theta_{j+1})$ have the only singularity at the point $(\varphi = \varphi_0,\ \theta_j = \theta_0)$, but it is just a logarithmic singularity that can be overcome by changing a variable in the next integration (see Section 3).

4.3 A Component of the Attraction Force along the Coordinate φ_0

For fixed values of the variables φ, φ_0, θ, θ_0, we denote

$$f(r) = \rho(\varphi,\theta,r)\,, \quad g(r) = \frac{\partial}{\partial\varphi_0}\frac{r^2}{\sqrt{r_0^2 - 2rr_0\cos\psi + r^2}} \ .$$

Then the function $G_0(x)$ in formula (5) has the form

$$G_0(x) = r_0\sin\theta\sin\theta_0\sin(\varphi-\varphi_0)\left[\mu_\varphi(x) + 3r_0\cos\psi\log(w(x))\right] \ ,$$

where

$$\mu_\varphi(x) = \frac{x^2 - 6r_0x\cos\psi + 3r_0^2}{sq(x)} + \frac{r_0x\cos\psi - r_0^2 + r_0sq(x)}{sq(x)\sin^2\psi} \ . \tag{11}$$

As in the previous subsection, there is a singularity in the second summand of the function $\mu_\varphi(x)$. As before, we apply quadrature (2) to compute the second integral. When calculating the first integral, we multiply the function $G_0(x)$ by the factor $\frac{\sin^2\psi}{\sin\theta\sin(\varphi-\varphi_0)}$ and denote an approximate value for the first integral that was calculated with such a factor for the fixed values φ and φ_0 by $I_2(\theta,\theta_0)$. Introducing the notation

$$f(\theta) = I_2(\theta,\theta_0) \ , \text{ and } \ g(\theta) = \frac{\sin^2\theta\sin(\varphi-\varphi_0)}{\sin^2\psi} \ ,$$

we can apply the quadrature (2) once again for computing the second integral. Having integrated the function $g(\theta)$ on every elementary interval $[\theta_j, \theta_{j+1}]$, we obtain

$$\begin{aligned} &\sin(\varphi-\varphi_0)\int\limits_{\theta_j}^{\theta_{j+1}} \tfrac{\sin^2\theta}{\sin^2\psi}\,d\theta \\ &= \tfrac{1}{\left(1-\sin^2\theta_0\sin^2\varphi_1\right)}\left[\mu_\varphi^{(1)}(\theta_j,\theta_{j+1})+\mu_\varphi^{(2)}(\theta_j,\theta_{j+1})+\mu_\varphi^{(3)}(\theta_j,\theta_{j+1})\right]\ , \end{aligned} \tag{12}$$

where $\varphi_1 = \varphi - \varphi_0$ and

$$\mu_\varphi^{(1)}(\theta_j,\theta_{j+1}) = \sin\theta_0\left(\cos^2\varphi_1 - \sin^2\varphi_1\cos^2\theta_0\right) sign\left(\sin(\varphi-\varphi_0)\right) v_1(\theta_j,\theta_{j+1})$$

$$\mu_\varphi^{(2)}(\theta_j,\theta_{j+1}) = \sin\varphi_1\sin\theta_0\cos\theta_0\cos\varphi_1\ v_2(\theta_j,\theta_{j+1})\ ,$$

$$\mu_\varphi^{(3)}(\theta_j,\theta_{j+1}) = \sin\varphi_1\left(\cos^2\theta_0 - \sin^2\theta_0\cos^2\varphi_1\right)(\theta_{j+1}-\theta_j)\ ,$$

and the functions $v_1(\theta_j,\theta_{j+1})$, $v_2(\theta_j,\theta_{j+1})$ were defined in the previous subsection. The integral (12) and the function $v_2(\theta_j,\theta_{j+1})$ have only the logarithmic singularity that is taken into account by changing variables in the next integration.

5 Analytical Computation of the Potential and the Attraction Force of an Ellipsoid for a Special Density Function

Let us consider an ellipsoid in the Cartesian coordinates with the largest semi-axis that is directed along the axis $\overrightarrow{OZ}$ and whose surface satisfies the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Let this ellipsoid have elliptic distribution of density that has a constant value on the surfaces of similar ellipsoids satisfying the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=k^2, \qquad k\in[0,\,1],$$

where $k=0$ corresponds to the center of the ellipsoid, and $k=1$ corresponds to the surface of the original ellipsoid. Then the density is a function of only one parameter k

$$\rho(M)=\rho(k)=\rho\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right).$$

5.1 The Potential of an Ellipsoid

The potential of an ellipsoid having such a density distribution (see [3]) is equal to

$$U(M_0)=\pi\,a\,b\,c\int\limits_{\lambda}^{\infty}\frac{\chi\left(k^2\right)}{R(s)}\,ds\ , \tag{13}$$

$$\chi\left(k^2\right)=\int\limits_{k^2}^{1}\rho(\alpha)\,d\alpha\ , \tag{14}$$

$$R(s) = \sqrt{(a^2+s)\,(b^2+s)\,(c^2+s)}\;, \tag{15}$$

$$k^2 = \frac{x_0^2}{a^2+s} + \frac{y_0^2}{b^2+s} + \frac{z_0^2}{c^2+s}\;, \tag{16}$$

$$\lambda = \begin{cases} 0, & M_0 \in T \\ \lambda_0, & M_0 \notin T\;, \end{cases} \tag{17}$$

and λ_0 satisfies the equation

$$\frac{x_0^2}{a^2+\lambda_0} + \frac{y_0^2}{b^2+\lambda_0} + \frac{z_0^2}{c^2+\lambda_0} = 1\;.$$

To illustrate the proposed numerical method for the computation of the potential, we consider an elongated ellipsoid of rotation around the axis $\overrightarrow{OZ}$ with the largest semi-axis of unit length and the other semi-axes that are equal to γ. The density function is chosen so that the integral (13) is computed analytically. Let

$$\rho(\alpha) = \frac{1}{(1+\alpha)^2}\;. \tag{18}$$

Let us turn to the spherical coordinates (φ,θ,r). Then from (18) it follows that the density of the ellipsoid at the point satisfying the equation $r^2\left(\frac{\sin^2\theta}{\gamma^2}+\cos^2\theta\right)=\alpha$, is $\frac{1}{(1+\alpha)^2}$. Then

$$\chi\left(k^2\right) = \frac{\left(1-k^2\right)}{2\left(1+k^2\right)}\;, \tag{19}$$

and from formulas (13), (15), (16), and (19), it follows, with allowance for the fact that for this case $a=b=\gamma$ and $c=1$,

$$U(M_0) = \frac{\pi\gamma^2}{2}\int_\lambda^\infty \upsilon(s)\,ds\;, \tag{20}$$

where

$$\upsilon(s) =$$

$$\frac{\left((\gamma^2+s)(1+s)-(1+s)\,r_0^2\sin^2\theta_0-(\gamma^2+s)r_0^2\cos^2\theta_0\right)}{\left((\gamma^2+s)(1+s)+(1+s)r_0^2\sin^2\theta_0+(\gamma^2+s)\,r_0^2\cos^2\theta_0\right)}\frac{1}{(\gamma^2+s)\sqrt{1+s}}\;.$$

The integral (20) can be computed analytically as

$$\begin{aligned}\frac{1}{\pi\gamma^2}U(\theta_0,r_0) &= \frac{q_+(s_2)}{(s_1-s_2)}\left[1-\frac{r_0^2\sin^2\theta_0}{\gamma_S}\left(s_1+\gamma^2\right)-\frac{r_0^2\cos^2\theta_0}{(1+s_2)}\right]\\ &+\frac{q_-(s_1)}{(s_1-s_2)}\left[-1+\frac{r_0^2\sin^2\theta_0}{\gamma_S}\left(s_2+\gamma^2\right)+\frac{r_0^2\cos^2\theta_0}{(1+s_1)}\right]+q_+\left(-\gamma^2\right)\frac{r_0^2\sin^2\theta_0}{\gamma_S}\;,\end{aligned} \tag{21}$$

where s_1 and s_2 are the smallest and the largest roots of the parabola $p_1(s)=s^2+s\left(1+\gamma^2+r_0^2\right)+\gamma^2+r_0^2\left(\sin^2\theta_0+\gamma^2\cos^2\theta_0\right)$, $s_1\in(-\infty,-1]$, $s_2\in[-1,0)$, $\gamma_S = s_1s_2+\gamma^2(s_1+s_2)+\gamma^4$,

$$q_-(s) = \sqrt{-1-s}\left(\frac{\pi}{2}-\arctan\left(\frac{\sqrt{1+\xi_0}}{\sqrt{-1-s}}\right)\right),$$

$$q_+(s) = \sqrt{1+s}\log\left(\frac{\sqrt{1+\xi_0}-\sqrt{1+s}}{\sqrt{\xi_0-s}}\right),$$

and ξ_0 is the largest root of the parabola

$$p_2(s) = s^2 + s\left(1 + \gamma^2 - r_0^2\right) + \gamma^2 - r_0^2\left(\sin^2\theta_0 + \gamma^2\cos^2\theta_0\right) .$$

In the formula (20), the lower limit of integration with respect to λ is defined by the formula (17), where λ_0 satisfies the equation

$$r_0^2\left(\frac{\sin^2\theta_0}{\gamma^2 + \lambda_0} + \frac{\cos^2\theta_0}{1 + \lambda_0}\right) = 1 .$$

5.2 Components of the Attraction Force of an Ellipsoid

Components of the attraction force of an ellipsoid having the elliptic density distribution $\rho\left(k^2\right)$ in the Cartesian coordinates are defined by formulas [3]

$$\begin{aligned} \frac{\partial U}{\partial x} &= -2\pi abcx \int_\lambda^\infty \frac{\rho\left(k^2\right) ds}{\left(a^2 + s\right) R(s)} , \\ \frac{\partial U}{\partial y} &= -2\pi abcy \int_\lambda^\infty \frac{\rho\left(k^2\right) ds}{\left(b^2 + s\right) R(s)} , \\ \frac{\partial U}{\partial z} &= -2\pi abcz \int_\lambda^\infty \frac{\rho\left(k^2\right) ds}{\left(c^2 + s\right) R(s)} , \end{aligned} \tag{22}$$

where the function $R(s)$ and λ were defined in (15) and (17). For the numerical experiments, as in the previous subsection, we consider an elongated ellipsoid of rotation with the semi-axes $a = b = \gamma$ and $c = 1$.

It should be noted that the analytic solution to the equations (22) does not exist for all the density functions $\rho(k^2)$. In particular, such a solution cannot be obtained for the density defined by the formula (18). For this reason, we choose such a density function that the equations (22) can be solved analytically,

$$\rho(\alpha) = \frac{1}{1 + \alpha} . \tag{23}$$

Let us turn to the spherical coordinates, where components of the attraction force of the ellipsoid of rotation with a density function defined by the formula (23) are

$$\frac{\partial U}{\partial r_0} = -2\pi\gamma^2 r_0\left[Q_1(\lambda, \gamma)\sin^2\theta_0 + Q_2(\lambda, \gamma)\cos^2\theta_0\right] , \tag{24}$$

$$\frac{\partial U}{\partial \theta_0} = -\pi\gamma^2 r_0^2 \sin(2\theta_0)\left[Q_1(\lambda, \gamma) - Q_2(\lambda, \gamma)\right] , \tag{25}$$

where

$$Q_1(\lambda, \gamma) = \int_\lambda^\infty \frac{\sqrt{1 + s}}{L(s)\left(\gamma^2 + s\right)} ds ,$$

$$Q_2(\lambda, \gamma) = \int_\lambda^\infty \frac{1}{L(s)\sqrt{1 + s}} ds ,$$

$$L(s) = \left(\gamma^2 + s\right)(1 + s) + r_0^2(1 + s)\sin^2\theta_0 + r_0^2\left(\gamma^2 + s\right)\cos^2\theta_0 .$$

Since the potential of an ellipsoid of rotation does not depend on the variable φ_0, then $\frac{\partial U}{\partial \varphi_0} = 0$.

The integrals $Q_1(\lambda, \gamma)$ and $Q_2(\lambda, \gamma)$ can be calculated analytically,

$$Q_1(\lambda,\gamma) =$$

$$\frac{2}{\gamma_S(s_1 - s_2)}\left(\left(s_1 + \gamma^2\right) q_+(s_2) - \left(s_2 + \gamma^2\right) q_-(s_1) - (s_1 - s_2)\, q_+\left(-\gamma^2\right)\right),$$

$$Q_2(\lambda,\gamma) = \frac{2}{(s_1 - s_2)}\left(\frac{q_+(s_2)}{(1 + s_2)} - \frac{q_-(s_1)}{(1 + s_1)}\right).$$

The functions $q_+(s)$ and $q_-(s)$, and the parameters s_1, s_2, and γ_S are defined in the previous section.

6 Numerical Experiments

In spherical coordinates, the potential of an ellipsoid of rotation is a function of only two coordinates, θ_0 and r_0.

To compute the integral (4) and its derivatives with respect to r_0 and θ_0, we choose a uniform grid along the coordinate r for points inside the body, a non-uniform grid for points outside the body, and uniform grids along the other coordinates. The upper limit in the inner integral for the ellipsoid of rotation is a function that depends only on the coordinate θ, $R(\theta) = \frac{\gamma}{\sqrt{\sin^2\theta + \gamma^2\cos^2\theta}}$. The density function ρ in (18) and (23) also is a function that depends only on the coordinates θ and r. Since the potential of an ellipsoid of rotation is a function that does not depend on the coordinate φ_0, then for taking into account a logarithmic singularity in the subsequent integration of the outer integrals, it is sufficient to do the change of variable $\varphi = t^2$.

We denote by $\varepsilon_{jk} = 100 \times \left|1 - \frac{\tilde{U}(\theta_j, r_k)}{U(\theta_j, r_k)}\right|$ the computation error (in percent) at the point θ_j, r_k for the approximated value of the potential $\tilde{U}(\theta_j, r_k)$, in which the inner integral in (4) was computed by the formula (2), with respect to the exact value of the potential at this point $U(\theta_j, r_k)$, computed by the formula (21). Values of the average $\varepsilon_{av} = \frac{1}{N_r N_\theta}\sum_{k=1}^{N_r}\sum_{j=1}^{N_\theta}\varepsilon_{jk}$ and the maximum $\varepsilon_{\max} = \max_{j,k}\varepsilon_{jk}$ errors for various numbers of the points N_r along the coordinate r are presented in Table 1. The number of points N_θ along the coordinate θ was $N_\theta = N_r$. The value γ in all the numerical experiments was 0.5, the number of points N_φ along the coordinate φ was 100, and the values r_0 varied in the interval from 0.001 up to 10. For each value θ_j, the number of nodes on the intervals $[0,\ R_\theta]$ and $[R_\theta, 10]$ was the same.

Table 1: Values of the average and maximum computation errors of potential in percent for various numbers of grid nodes.

N_r	50	100	200	400
ε_{av}	0.1331	0.0337	0.0090	0.0027
$\varepsilon_{\max}$	0.4835	0.1361	0.0379	0.0105

To illustrate the numerical computation of components of the attraction force of an ellipsoid, we choose another error because the components calculated by the formulas (24) and (25) can be too small in modulus at a few points of the domain.

We denote

$$\delta_{av}^{(r)} = \sum_{k=1}^{N_r}\sum_{j=1}^{N_\theta}\left(\frac{\partial \tilde{U}}{\partial r_0}(\theta_j, r_k) - \frac{\partial U}{\partial r_0}(\theta_j, r_k)\right)^2 \Bigg/ \sum_{k=1}^{N_r}\sum_{j=1}^{N_\theta}\left(\frac{\partial U}{\partial r_0}(\theta_j, r_k)\right)^2,$$

$$\delta_{av}^{(\theta)} = \sum_{k=1}^{N_r}\sum_{j=1}^{N_\theta}\left(\frac{\partial \tilde{U}}{\partial \theta_0}(\theta_j, r_k) - \frac{\partial U}{\partial \theta_0}(\theta_j, r_k)\right)^2 \Bigg/ \sum_{k=1}^{N_r}\sum_{j=1}^{N_\theta}\left(\frac{\partial U}{\partial \theta_0}(\theta_j, r_k)\right)^2,$$

$$\delta_{\max}^{(r)} = \max_{k,j}\left|\frac{\partial \tilde{U}}{\partial r_0}(\theta_j, r_k) - \frac{\partial U}{\partial r_0}(\theta_j, r_k)\right|,$$

$$\delta_{\max}^{(\theta)} = \max_{k,j}\left|\frac{\partial \tilde{U}}{\partial \theta_0}(\theta_j, r_k) - \frac{\partial U}{\partial \theta_0}(\theta_j, r_k)\right|.$$

Table 2 presents values of errors for the computation of force components for the same values of numerical parameters as well as for the computation of the potential.

Table 2: Values of the average and the maximum computation errors of the force components in percent for various numbers of grid nodes.

N_r	50	100	200	400
$\delta_{av}^{(r)}$	0.726 E-5	0.715 E-6	0.852 E-7	0.139 E-7
$\delta_{\max}^{(r)}$	0.189 E-2	0.935 E-3	0.466 E-3	0.233 E-3
$\delta_{av}^{(\theta)}$	0.844 E-4	0.137 E-4	0.251 E-5	0.480 E-6
$\delta_{\max}^{(\theta)}$	0.929 E-3	0.450 E-3	0.220 E-3	0.108 E-3

When computing the force component along the coordinate φ_0, we consider the potential $U(\varphi_0, \theta_0, r_0)$ as a function of three variables for the ellipsoid of rotation. Let

$$\delta_{av}^{(\varphi)} = \sum_{i=1}^{N_\varphi}\sum_{k=1}^{N_r}\sum_{j=1}^{N_\theta}\left|\frac{\partial \tilde{U}}{\partial \varphi_0}(\varphi_i, \theta_j, r_k)\right|, \quad \delta_{\max}^{(\varphi)} = \max_{k,j,i}\left|\frac{\partial \tilde{U}}{\partial \varphi_0}(\varphi_i, \theta_j, r_k)\right|.$$

Then $\delta_{av}^{(\varphi)} \approx 0.2 \times 10^{-10}$ and $\delta_{\max}^{(\varphi)} \approx 0.4 \times 10^{-9}$ for all the numbers N_r.

Values of the errors presented in Tables 1 and 2 reveal increasing the accuracy when doubling the number of nodes along the coordinates r and θ

- in the range from 3.33 to 4 times when the potential is computed,
- in the range from 5 to 10 for the average error of components of the force,
- approximately at a twofold rate for a maximum error of components of the force.

The results of the tests presented allow us to conclude that the numerical approbation of our method proposed for computing the volume potential and attraction force of an ellipsoid is successful. To attain a higher accuracy, it is sufficient to increase the number of the discretization nodes. This circumstance is not valid in the application of many other methods because of the presence of singularity in the integrand.

Our concept for the calculation of the potential and the attraction force of an ellipsoid can be applied to the bodies having other forms.

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