# Generalizations of Theorems of Rohn and Vrahatis *

G. Heindl
Fachbereich Mathematik und Naturwissenschaften,
Universität Wuppertal, Germany
Gerhard.Heindl@math.uni-wuppertal.de

**Abstract**

An existence theorem for solutions of systems of nonlinear equations proved by J. Rohn is extended in two directions, one with the aim to cover a generalized version of the well-known Theorem of Miranda, stated but not proved by R.E. Moore and J.B. Kioustelidis, the other with the aim to eliminate the dependence of Rohn's Theorem on the standard basis of $\mathbb{R}^n$. The resulting theorem also extends a generalized version of Miranda's Theorem proved by M.N. Vrahatis.

**Keywords:** Inclusion of zeros of nonlinear mappings.
**AMS subject classifications:** 65G40

## 1 The Considered Theorem of Rohn

We will be concerned with an existence theorem for solutions of systems of nonlinear equations presented by J. Rohn [11].
Given a nonempty convex compact set $C \subset \mathbb{R}^n$, Rohn introduced the subsets

$$\begin{array}{rcl} C_j^- & := & \{x \in C : x - te_j \notin C \ \forall\ t > 0\}, \\ C_j^+ & := & \{x \in C : x + te_j \notin C \ \forall\ t > 0\}, \\ & & j = 1, \ldots, n, \end{array}$$

where $e_1, \ldots, e_n$ denote the standard unit vectors in $\mathbb{R}^n$, and proved

**Theorem 1.1** *Let $F : C \to \mathbb{R}^n$ be a continuous mapping satisfying the condition*

$$(R) \qquad \begin{cases} F(x)_j \leq 0 \ \forall\ x \in C_j^-, \\ F(y)_j \geq 0 \ \forall\ y \in C_j^+, \\ j = 1, \ldots, n. \end{cases}$$

*Then there is a $z \in C$ such that $F(z) = 0$.*

*Submitted: June 1, 2016; Accepted. September 30, 2016.

Rohn's Theorem extends the well-known Theorem of Miranda [9] which is identical with Theorem 1.1 (for $-F$) when $C$ is an n-dimensional cube

$$\{x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n : |x_j| \le L, j = 1, \ldots, n\}.$$

# 2 Relaxing $(R)$

An obvious extension of Theorem 1.1 is obtained by observing that (R) can be replaced by

$$(R_{+-}) \qquad \begin{cases} F(x)_j \le 0 \;\forall\; x \in C_j^- \text{ and } F(y)_j \ge 0 \;\forall\; y \in C_j^+ & (\text{ j +}) \\ \text{or} & \\ F(x)_j \ge 0 \;\forall\; x \in C_j^- \text{ and } F(y)_j \le 0 \;\forall\; y \in C_j^+ & (\text{ j - }), \\ j = 1, \ldots, n. & \end{cases}$$

*Proof:* Assuming that $(R_{+-})$ holds, we set $s_j = e_j$ if $(j+)$ holds , $s_j = -e_j$ if $(j-)$ holds, $j = 1, \ldots, n$. Then $(R_{+-})$ corresponds to $(R)$ for the scaled mapping $\hat{F} = S * F$ , where $S$ denotes the nonsingular matrix $(s_1, \ldots, s_n)^T$. Since $F$ has the same zeros as $\hat{F}$, the proof is complete. □

If $C$ is an n-dimensional (possibly degenerated) interval vector

$$[a, b] := \{x \in \mathbb{R}^n : a_j \le x_j \le b_j, j = 1, \ldots, n\},\; a, b \in \mathbb{R}^n : a_j \le b_j, j = 1, \ldots, n,$$

then $(R_{+-})$ often is wrongly interpreted as

$$(R') \qquad \begin{cases} F(x)_j F(y)_j \le 0 \;\forall\; x \in C_j^-,\; y \in C_j^+, \\ j = 1, \ldots, n. \end{cases}$$

See e.g. the comments afterTheorem 1 in [10] and afterTheorem 1.3 in [8]. The authors of these papers claim that $(R')$ means that the $j$ th component of $F$ has opposite signs on $C_j^-$ and $C_j^+, j = 1, \ldots, n$. This however must not be the case. If e.g. $F(x)_j$ is identically zero on $C_j^-$, then $F(y)_j$ can change between "$< 0$ ", "$= 0$ " and "$> 0$ " on $C_j^+$. Obviously $(R_{+-})$ implies $(R')$, however $(R_{+-})$ usually is not a consequence of $(R')$ as the following example shows:

**Example 2.1**

$$C = [a, b] \subset \mathbb{R}^2 \textit{with } a = (-1, -1)^T, b = (1, 1)^T, F(x)_1 = x_1 + \varepsilon, -1 \le \varepsilon \le 1,$$
$$F(x)_2 = x_1(x_2 + 1).$$

*($(R')$ holds since $F(x)_2 = 0$ if $x \in [a, b]_2^-, F(x)_1 = -1 + \varepsilon \le 0$ if $x \in [a, b]_1^-$, and $F(y)_1 = 1 + \varepsilon \ge 0$ if $y \in [a, b]_1^+$. However $(R_{+-})$ does not hold since $(-1, 1)^T$ and $(1, 1)^T$ are both in $[a, b]_2^+$ but $F((-1, 1)^T)_2 = -2$ and $F((1, 1)^T)_2 = 2$.)*

Consequently the cited theorems in [10] and [8] are not obvious extensions of Miranda's Theorem [9]. Nevertheless they are valid. This will be shown by proving

**Proposition 2.1** *Let $[a, b]$ be a (possibly degenerated) $n$-dimensional interval vector, and $F : [a, b] \to \mathbb{R}^n$ a continuous mapping satisfying the condition*

$$(M') \qquad \begin{cases} F(x)_j F(y)_j \le 0 \; \forall \; x \in [a, b]_j^-, \; y \in [a, b]_j^+, \\ j = 1, \ldots, n. \end{cases}$$

*Then there is a $z \in [a, b]$ such that $F(z) = 0$.*

*Proof:* The proof will be given by induction on the number of indices $j$ such that $a_j < b_j$, i.e on the dimension $dim[a, b]$ of $[a, b]$ as a convex subset of $\mathbb{R}^n$. Let us first note that $[a, b]_j^- = \{x \in [a, b] : x_j = a_j\}$ and $[a, b]_j^+ = \{x \in [a, b] : x_j = b_j\}$ are interval vectors too. If $dim[a, b] = 0$ then $[a, b]_j^- = [a, b]_j^+ = \{a\}, j = 1, \ldots, n$. Hence, by $(M')$ $F(a)_j^2 \le 0, \; j = 1, \ldots, n,$ hence $F(a) = 0$.
Now assume $dim[a, b] = k \in \{1, \ldots, n\}$, and that Proposition 1 holds whenever $[a, b]$ is replaced by an interval $[c, d]$ with $dim[c, d] < k$.
Let $J$ denote the nonempty set of indices $j \in \{1, \ldots, n\}$ such that $a_j < b_j$.
$i \in \{1, \ldots, n\} \backslash J$ holds iff $[a, b]_i^- = [a, b]_i^+ = [a, b]$.
Hence $(M')$ implies $F(x)_i = 0 \; \forall \; x \in [a, b]$ if $i \notin J$.
Case 1: There is an index $i \in J$ such that $F(x)_i = 0 \; \forall \; x \in [a, b]_i^-$.
Then for $[c, d] := [a, b]_i^-$ we have $[c, d]_i^- = [c, d]_i^+ = [c, d]$ and
$\forall j \in \{1, \ldots, n\} \backslash \{i\}$ :
$[c, d]_j^- = [a, b]_i^- \cap [a, b]_j^- \subseteq [a, b]_j^-$, $[c, d]_j^+ = [a, b]_i^- \cap [a, b]_j^+ \subseteq [a, b]_j^+$. Hence $(M')$ holds also if $[a, b]$ is replaced by $[a, b]_i^-$. Since $dim[a, b]_i^- = k - 1$ we can conclude that there is a zero of $F$ in $[a, b]_i^-$.
Case 2: There is an index $i \in J$ such that $F(x)_i = 0 \; \forall \; x \in [a, b]_i^+$.
Then we can conclude in the same way as in Case 1 that there is a zero of $F$ in $[a, b]_i^+$.
Case 3: For every $j \in J$ there is an $x^j \in [a, b]_j^-$ such that $F(x^j)_j \neq 0$, and a $y^j \in [a, b]_j^+$ such that $F(y^j)_j \neq 0$. Together with $(M')$ this implies

$$F(x^j)_j F(y^j)_j < 0, \text{ i.e. } sign(F(y^j)_j) = -sign(F(x^j)_j) \neq 0 \; \forall \; j \in J.$$

If for $j \in J$ $F(x^j)_j < 0$ and $F(y^j)_j > 0$ then $(M')$ implies

$$F(x)_j \le 0 \; \forall \; x \in [a, b]_j^- \text{ and } F(y)_j \ge 0 \; \forall \; y \in [a, b]_j^+.$$

If for $j \in J$ $F(x^j)_j > 0$ and $F(y^j)_j < 0$ then $(M')$ implies

$$F(x)_j \ge 0 \; \forall \; x \in [a, b]_j^- \text{ and } F(y)_j \le 0 \; \forall \; y \in [a, b]_j^+.$$

For all $j \in \{1, \ldots, n\} \backslash J$ we have $F(x)_j = 0 \; \forall \; x \in [a, b]$.
Hence $F$ satisfies $(R_{+-})$ for $C = [a, b]$, showing that $F$ has a zero in $[a, b]$. $\square$

Proposition 2.1 will be extended from $[a, b]$ to an arbitrary convex compact set $C \subseteq \mathbb{R}^n$ by making use of a special retraction of $\mathbb{R}^n$ onto $C$, which will be introduced now.
Let $\|.\|$ denote the euclidean norm on $\mathbb{R}^n$. Then for every $x \in \mathbb{R}^n$ there is a unique $\bar{x} \in C$ such that

$$\|x - \bar{x}\| = min\{\|x - y\| : y \in C\}.$$

The mapping $p_C : \mathbb{R}^n \ni x \mapsto \bar{x} \in C$ is the orthogonal projection of $\mathbb{R}^n$ onto $C$. It is Lipschitz continuous with Lipschitz constant 1. (See e.g. [7] Proposition (2.2.4)).
In addition for any nonzero $e \in \mathbb{R}^n$ we consider the (nonempty) subset

$$C_e := \{x \in C : x + te \notin C \; \forall \; t > 0\}$$

of $C$ which is compact in case $n \leq 2$ but not necessarily if $n \geq 3$. If $(b_1, \ldots, b_n)$ is a basis of $\mathbb{R}^n$ then for every $x \in \partial C$ there is a $j \in \{1, \ldots, n\}$ such that $x \in C_{b_j} \cup C_{-b_j}$. $C_{-e_j} = C_j^-$ , $C_{e_j} = C_j^+$.

**Proposition 2.2** *Let $x \in \mathbb{R}^n \backslash C$ and $e \in \mathbb{R}^n \backslash \{0\}$ satisfy $e^T(\bar{x} - x) < 0$. Then $\bar{x} \in C_e$.*

*Proof:* It is well-known that

$$(y - \bar{x})^T(\bar{x} - x) \geq 0 \ \forall \ y \in C.$$

(See e.g. the proof of Theorem 3 in Appendix 1 of [6] ). Hence $\bar{x} + te \in C$ implies $te^T(\bar{x} - x) \geq 0$, and $e^T(\bar{x} - x) < 0$ implies $t \leq 0$. $\square$

Now we can prove

**Theorem 2.1** *Let $C$ be a nonempty convex compact subset of $\mathbb{R}^n$ and $F : C \to \mathbb{R}^n$ a continuous mapping satisfying condition $(R')$.*
*Then there is a $z \in C$ such that $F(z) = 0$.*

*Proof:* Since $C$ is bounded, there is an $L > 0$ such that $|x_i| < L$, $i = 1, \ldots, n$, for all $x \in C$. Let $a, b \in \mathbb{R}^n$ be defined by $a_i := -L, b_i := L$, $i = 1, \ldots, n$. Then
$[a, b]_j^- = \{x \in [a, b] : x_j = -L\}$, $[a, b]_j^+ = \{x \in [a, b] : x_j = L\}$, $j = 1, \ldots, n$.
If $x \in [a, b]_j^-$ then $x \in \mathbb{R}^n \backslash C$ and $-e_j^T(\bar{x} - x) = -\bar{x}_j + x_j = -\bar{x}_j - L < 0$.
Hence $\bar{x} \in C_{-e_j}$ by Proposition 2.2. Consequently $range(p_C, [a, b]_j^-) \subseteq C_{-e_j}$ and similarly $range(p_C, [a, b]_j^+) \subseteq C_{e_j}$. Now we show that the existence of a zero of $F$ can be concluded even from the weaker condition

$$(R_r') \ F(u)_j F(v)_j \leq 0 \ \forall \ u \in range(p_C, [a, b]_j^-), v \in range(p_C, ([a, b]_j^+), j = 1, \ldots, n.$$

In fact $(R_r')$ means that the continuous mapping $G := F \circ p_C$ satisfies $(M')$ with $F$ replaced by $G$.
Consequently there is an $x \in [a, b]$ such that $G(x) = 0$. Since $G(x) = F(\bar{x})$, and $z := \bar{x} \in C$, Theorem 2.1 is proved. $\square$

**Remarks 2.1** *1. $(R_r')$ is sometimes a much weaker condition than $(R')$. If e.g. $n = 2$, $C = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ and $r = 2$, then $C_1^- = \{x \in \mathbb{R}^2 : \|x\| = 1$ and $x_1 \leq 0\}$, whereas $p_C([a, b]_1^-) = \{x \in C_1^- : x_1 \leq -1/\sqrt{2}\}$. Equivalent relations hold for the other faces of $[a, b]$.*
*2. The proof of Theorem 2.1 shows up also an alternative way to prove Rohn's original Theorem 1.1*
*3. The mapping $g : I \ni x \mapsto p_C(Lx) \in C$, where $I$ denotes the unit-cube $\{x = (x_1, \ldots, x_n)^T : |x_j| \leq 1, j = 1, \ldots, n\}$, is a Miranda mapping for $C$ in the sense of J. Mayer ([8] Definition 2.1), i.e. $g$ is continuous, $range(g, I) = C$ and $range(g, \partial I) = \partial C$. (For showing $range(g, \partial I) = \partial C$ , note that for all $x \in \partial C$ there is an $e \in \mathbb{R}^n \backslash \{0\}$ such that $x \in C_e$ and $x = p_C(x + te)$ for all $t > 0$. (See e.g. the proof of Theorem 4 in Appendix 1 of [6])). Since $range(g, I_j^-) = range(p_C, [a, b]_j^-), range(g, I_j^+) = range(p_C, [a, b]_j^+), j = 1, \ldots, n$, the proof of Theorem 2.1 can be considered also as a special application of Theorem 2.7 in [8] (now verified by Proposition 2.1) which states that there is a zero of $F$ in $C$ if*

$$F(u)_j F(v)_j \leq 0 \ \forall \ u \in range(g, I_j^-) \ , \ v \in range(g, I_j^+).$$

# 3 The Final Generalization

The aim of the final generalization of Theorem 1.1 is to eliminate the dependence of Theorem 2.1 on the standard basis of $\mathbb{R}^n$. In fact it can be shown that Theorem 2.1 remains valid if the sets $C_j^-, C_j^+$ and the components of $F$ are defined with respect to arbitrary, even different, bases of $\mathbb{R}^n$. As a consequence, the resulting theorem extends also a generalization of Miranda's Theorem due to M. Vrahatis ([13], Theorem 2).
The main part of this theorem can be described as follows:

Let $(b_1, \ldots, b_n)$ be a basis of $\mathbb{R}^n$, $L > 0$,
$C := \{x \in \mathbb{R}^n : |b_j^T x| \leq L, j = 1, \ldots, n\}$, and $F : C \to \mathbb{R}^n$ a continuous mapping such that $F(x) \neq 0$ for $x \in \partial C$ and

$$(V) \qquad \begin{cases} b_j^T F(x) \geq 0 \text{ if } x \in C \text{ and } b_j^T x = -L, \\ b_j^T F(y) \leq 0 \text{ if } y \in C \text{ and } b_j^T y = L, \\ j = 1, \ldots, n. \end{cases}$$

Then $F(x) = 0$ has a solution in $C^0$.

As a preparatory step we need

**Proposition 3.1** *Let $B$ be a real nonsingular $(n,n)$ - matrix, $d \in \mathbb{R}^n$,
$h : \mathbb{R}^n \ni x \mapsto Bx + d \in \mathbb{R}^n$, and $\tilde{C}$ a nonempty convex compact subset of $\mathbb{R}^n$. Then $C := range(h, \tilde{C})$ is also a nonempty convex compact set and $range(h, \tilde{C}_e) = C_{Be}$ holds for all $e \in \mathbb{R}^n \backslash \{0\}$.*

*Proof:* It is easily shown that the assumptions $range(h, \tilde{C}_e) \not\subseteq C_{Be}$
and $C_{Be} \not\subseteq range(h, \tilde{C}_e)$ lead to contradictions. $\square$

Now we consider two nonempty subsets $C, D$ of $\mathbb{R}^n$, $C$ convex and compact, and two bases $(b_1, \ldots, b_n), (s_1, \ldots, s_n)$ of $\mathbb{R}^n$.Then we can prove

**Theorem 3.1** *Let $g : C \to D$ and $F : D \to \mathbb{R}^n$ be continuous mappings satisfying the condition*

$$(R'') \; (s_j^T F(u))(s_j^T F(v)) \leq 0 \; \forall \; u \in range(g, C_{-b_j}), v \in range(g, C_{b_j}), j = 1, \ldots, n.$$

*Then there is a $z \in C$ such that $F(z) = 0$.*

*Proof:* We introduce the nonsingular matrix $B$ with columns $b_1, \ldots, b_n$, the corresponding linear mapping $h : \mathbb{R}^n \ni x \mapsto Bx \in \mathbb{R}^n$ and $\tilde{C} := B^{-1}C$.
Then $\tilde{C}$ is a convex compact subset of $\mathbb{R}^n$ such that $range(h, \tilde{C}) = C$ and (by Proposition 3.1)

$$range(h, \tilde{C}_{e_j}) = C_{Be_j} = C_{b_j}, range(h, \tilde{C}_{-e_j}) = C_{B(-e_j)} = C_{-b_j}, j = 1, \ldots, n.$$

Now let $\tilde{F}$ be the continuous mapping $\tilde{C} \ni x \mapsto S * F(g(h(x))) \in \mathbb{R}^n$, where S denotes the nonsingular scaling matrix $(s_1, \ldots, s_n)^T$. Then $\tilde{F}$ satisfies the conditions

$$\tilde{F}(x)_j \tilde{F}(y)_j \leq 0 \; \forall \; x \in \tilde{C}_{-e_j}, y \in \tilde{C}_{e_j}, j = 1, \ldots, n.$$

Hence by Theorem 2.1 there is a $\tilde{z} \in \tilde{C} : \tilde{F}(\tilde{z}) = 0$. Since $S$ is nonsingular, $z := g(h(\tilde{z}))$ is a zero of $F$. $\square$

The announced generalization of Theorem 2 in [13] is given by the

**Corollary 3.1** *Let* $(b_1,\dots,b_n),(s_1,\dots,s_n)$ *be bases of* $\mathbb{R}^n, x_0 \in \mathbb{R}^n$,
$L_i \geq 0,\ i = 1\dots,n\ ,\ C := \{x \in \mathbb{R}^n : |b_i^T(x-x_0)| \leq L_i, i = 1,\dots,n\}$,
$F : C \to \mathbb{R}^n$ *a continuous mapping with the property*
$(s_j^T F(x))(s_j^T F(y)) \leq 0$ *if* $x \in C$ *satisfies* $b_j^T(x-x_0) = -L_j$ *and* $y \in C$ *satisfies* $b_j^T(y-x_0) = L_j, j = 1,\dots,n$.
*Then there is a* $z \in C$ *such that* $F(z) = 0$.

*Proof:* $C$ is a convex compact subset of $\mathbb{R}^n$ with $x_0 \in C$.
Let $(b_1^*,\dots,b_n^*)$ denote the dual basis of $(b_1,\dots,b_n)$,
(i.e. $b_i^T b_j^* = \delta_{ij}, i,j = 1,\dots,n$). Then for all $j \in \{1,\dots,n\}$
$C_{-b_j^*} = \{x \in C : b_j^T(x-x_0) = -L_j\}$, $C_{b_j^*} = \{x \in C : b_j^T(x-x_0) = L_j\}$ holds. Hence the conditions of Theorem 3.1 are satisfied with $b_j$ replaced by $b_j^*, j = 1,\dots,n$, $D = C$ and $g = id|C$. $\square$

The following simple example illustrates that there are situations where the corollary is applicable but the considered theorems of Rohn and Vrahatis are not.

**Example 3.1** $C := \{x \in \mathbb{R}^2 : |b_i^T x| \leq 4, i = 1,2\}$, *where* $b_1^T := (-1,2)$,
$b_2^T := (3,1)$. $F(x)_1 := b_1^T x - 3\ ,\ F(x)_2 := b_2^T x + 2\ \forall\ x \in C$.

*( $C$ is the convex hull of the points*
$p_1 = (12,-8)^T/7, p_2 = (4,16)^T/7, p_3 = -p1, p_4 = -p_2$.
$F(p_1) = (-7,6)^T, F(p_2) = (1,6)^T, F(p_3) = (1,-2)^T, F(p_4) = (-7,-2)^T$.
$C_1^-$ *is the union of the segments* $< p_2,p_3 >$ *and* $< p_3,p_4 >$.
$F(p_3)_1 = 1, F(p_4)_1 = -7$, *hence* $(R)$ *is not satisfied.*
$F$ *has no zero on the boundary of* $C$.
$\{x \in C : b_2^T x = -4\}$ *is the segment* $< p_3,p_4 >$.
$b_2^T F(p_3) = 1, b_2^T F(p_4) = -23$. *Hence* $(V)$ *is not satisfied.*
$\{x \in C : b_1^T x = -4\} =< p_4,p_1 >, \{y \in C : b_1^T y = 4\} =< p_2.p_3 >$.
$\{x \in C : b_2^T x = -4\} =< p_3,p_4 >, \{y \in C : b_2^T y = 4\} =< p_1,p_2 >$.
$F(x)_1 F(y)_1 = -7$ *if* $x \in< p_4,p_1 >, y \in< p_2,p_3 >$,
$F(x)_2 F(y)_2 = -12$ *if* $x \in< p_3,p_4 >, y \in< p_1,p_2 >$.
*Hence the conditions of Corollary 3.1 hold with* $x_0 = 0$
*and* $(s_1,s_2) = (e_1,e_2)$*).*

**Remarks 3.1** *concerning the choice of* $C$ *and* $g$ *:*
*In practical examples one usually wants to test whether there is a zero of* $F$ *in a given (closed) norm ball* $D' \subseteq D$. *Say* $D'$ *is defined with respect to the norm* $p$, *has radius* $r'$ *and center* $x_0$. *Then* $C$ *should be chosen also as a norm ball, say with respect to the norm* $q$, *with the same center as* $D'$ *and such that* $D' \subseteq int(C)$. *Further* $g$ *should be chosen such that* $range(g,C) = D'$, $range(g,\partial C) = \partial D'$ *and* $range(g,C_{-b}) = range(r \circ g, C_b))$ *for all* $b \in \mathbb{R}^n\backslash\{0\}$, *where* $r$ *denotes the reflection on* $x_0$ *defined by* $r(x) = 2x_0 - x\ \forall\ x \in \mathbb{R}^n$. *A well-known mapping with these properties is the retraction of* $\mathbb{R}^n$ *onto* $D'$ *given by*

$$g(x) = \begin{cases} x \ \ if\ x \in D', \\ x_0 + r'\frac{x-x_0}{p(x-x_0)} \ \ if\ x \in C\backslash D'. \end{cases}$$

*However also* $g = p_{D'}|C$ *can be used. Note that both mappings are continuous and that* $r \circ g = g \circ r|C$ *holds.*

Let us consider an example which was treated by J. Mayer in [8] by a test based on his Theorem 2.7 (which results from Theorem 3.1 if $C = I$, the standard unit cube in $\mathbb{R}^n$, $b_j = s_j = e_j, j = 1 \dots n$, and $g$ a Miranda mapping from $I$ onto $D'$).

**Example 3.2** $D' = \{x \in \mathbb{R}^2 : ||x - (1/2, 1/2)^T|| \leq 1/2\}$
*($||.||$ denoting the euclidean norm on $\mathbb{R}^2$).*
*$F$ is defined for $x = (x_1, x_2)^T \in \mathbb{R}^2 = D$ by*
$F(x) = ((x_1^2 + x_2^2)^2 - 2x_1, (x_1 - 1)^2 + (x_2 - 1)^2 - 1)^T$.
*Applying Theorem 3.1 with $C = \{x \in \mathbb{R}^2 : |x_1 - 1/2| + |x_2 - 1/2| \leq 1\}$, $g = p_{D'}|C$, and $b_1 = (-1, 1)^T, b_2 = -(1, 1)^T, s_1 = (1, 0)^T, s_2 = (0, 1)^T$, we get*
$range(g, C_{b_1}) = \{x \in \partial D' : x_1 \leq 1/2, x_2 \geq 1/2\}$,
$range(g, C_{-b_1}) = \{x \in \partial D' : x_1 \geq 1/2, x_2 \leq 1/2\} = range(r \circ g, C_{b_1})$,
$range(gC_{b_2}) = \{x \in \partial D' : x_1 \leq 1/2, x_2 \leq 1/2\}$,
$range(g, C_{-b_2}) = \{x \in \partial D' : x_1 \geq 1/2, x_2 \geq 1/2\} = range(r \circ g, C_{b_2})$.
*(These sets constitute the Miranda partition used by J. Mayer).*
*Since $\partial D' = \{(1 + cos(\varphi), 1 + sin(\varphi))^T/2 : -\pi < \varphi \leq \pi\}$ it is easily shown :*
$s_1^T F(x) \geq 1/16$ *if* $x \in range(g, C_{b_1})$,
$s_1^T F(y) \leq -7/16$ *if* $y \in range(g, C_{-b_1})$,
$s_2^T F(x) \geq 1/4$ *if* $x \in range(g, C_{b_2})$,
$s_2^T F(y) \leq -3/4$ *if* $y \in range(g, C_{-b_2})$.
*Hence Theorem 3.1 applies*

# 4 Computational Aspects

In verification methods based on Miranda's Theorem or one of its extensions, it is necessary to test inequalities of type " $\leq$ " (or " $\geq$ "). Since however in most cases of real computations rounding errors must be included, " $\leq$ " (" $\geq$ ") can be verified usually only if strict inequalities hold.
Let us assume therefore that $(R'')$ holds with " $\leq$ " replaced by " $<$ " and in addition that the conditions described in the last remark are satisfied.
Then we have especially

$$(R_<) \qquad (s_j^T F(r(x)))(s_j^T F(x)) < 0 \;\forall\; x \in range(g, C_{b_j}), j = 1, \dots, n.$$

But this implies the scaling invariant property

$$(B) \qquad F(x) \neq 0 \text{ and } F(r(x)) \neq \lambda F(x) \;\forall\; x \in \partial D' \text{ and } \lambda > 0,$$

as can be shown as follows: For any $x \in \partial D'$ there is a $j \in \{1, \dots, n\}$ such that $x \in range(g, C_{b_j})$ or $x \in range(g, C_{-b_j}) = range(r \circ g, C_{b_j})$. In any case we have $F(x) \neq 0$. Assume $F(r(x)) = \lambda F(x)$. Then in case $x \in range(g, C_{b_j})$ we have $0 > (c_j^T F(r(x)))(c_j^T F(x)) = \lambda (c_j^T F(x))^2$, hence $\lambda < 0$. Similarly this follows also in case $x \in range(r \circ g, C_{b_j})$ i.e. $r(x) \in range(g, C_{b_j})$.
By a corollary of a well-known theorem of Borsuk published in [2], see e.g. [3] Corollary 4.1, $(B)$ is a sufficient condition for the existence of a zero of $F$ in $int(D')$. In [1] $(B)$ was compared with the conditions of an affine invariant version of the existence theorem of Kantorovich [4] and an extended Miranda theorem. In [5] $(B)$ was used as a basis for checking the equivalent condition

$$(B') \qquad F(x) \neq 0 \text{ and } \frac{F(r(x))^T F(x)}{||F(r(x))||||F(x)||} < 1 \;\forall\; x \in \partial D'$$

for verifying the existence of zeros of continuous mappings from interval vectors in $\mathbb{R}^n$ to $\mathbb{R}^n$ by interval arithmetic methods. $(B')$ can be checked however also very flexible by interval arithmetic methods if $D'$ is not an interval vector. Let us test $(B')$ e.g. in Example 3.2 : For every $j = 1, \dots, 200$ let $J_j$ denote an outward rounded version of the interval $[(j-1)\pi/200, j\pi/200]$.
Then with $x(\varphi) := (1 + cos(\varphi), 1 + sin(\varphi))^T/2$ , $\varphi \in\ ]-\pi, \pi]$, we have $r(x(\varphi)) = (1-cos(\varphi), 1-sin(\varphi))^T/2$ and by a simple verifying program using INTLAB [12] we get $F(x(\varphi)) \neq 0$ and

$$\frac{F(r(x(\varphi)))^T F(x(\varphi))}{||F(r(x(\varphi)))||||F(x(\varphi))||} < -0.494\ \forall\ \varphi \in J_j, j = 1, \dots, 200.$$

# References


[1] Götz Alefeld, Andreas Frommer, Gerhard Heindl, and Jan Mayer. On the existence theorems of Kantorovich, Miranda and Borsuk. *ETNA*, 17:102–111, 2004.

[2] Karol Borsuk. Drei Sätze über die n-dimensionale Sphäre. *Fund. Math.*, 20:177–190, 1933.

[3] Klaus Deimling. *Nonlinear Functional Analysis.* Springer, Berlin, Heidelberg, New York, 1985.

[4] Peter Deuflhard and Gerhard Heindl. Affine invariant convergence theorems for Newton's method and extensions to related methods. *SIAM J. Numer. Anal.*, 16:1–10, 1980.

[5] Andreas Frommer and Bruno Lang. Existence tests for solutions of nonlinear equations using Borsuk's theorem. *SIAM J. Numer. Anal.*, 43(3):1348–1361, 2005.

[6] Serge Lang. *Linear Algebra.* Addison-Wesley Publishing Company, Massachusetts, London, Manila, second edition edition, 1972.

[7] Pierre-Jean Laurent. *Approximation et optimisation.* Number 13 in Collection Enseignement des sciences. Hermann, Paris, 1972.

[8] Jan Mayer. A generalized theorem of Miranda and the theorem of Newton-Kantorovich. *Numer. Funct. Anal. Optimization*, 23(3-4):333–357, 2002.

[9] Carlo Miranda. Un' osservazione su un teorema di Brouwer. *Bull. Unione Mat. Ital.*, Ser. 2(3):5–7, 1940.

[10] Ramon B. Moore and John B. Kioustelidis. A simple test for accuracy of approximate solutions to nonlinear (or linear) systems. *SIAM J. Numer. Math.*, 17(4):521–529, 1980.

[11] Jiří Rohn. An existence theorem for systems of nonlinear equations. *ZAMM*, 60:345, 1980.

[12] Siegfried Rump. INTLAB - INTerval LABoratory. In Tibor Csendes, editor, *Developments in Reliable Computing*, pages 77–104, Dordrecht, 1999. Kluwer Academic Publishers.

[13] Michael N. Vrahatis. A short proof and a generalization of Miranda's existence theorem. *Proc. of the AMS*, 107(3):701–703, 1989.