A Fixed Point Theorem Based on a Modified Midpoint – Radius Interval Arithmetic*

dedicated to U. Kulisch on the occasion of his 85th birthday

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Abstract

In this paper we extend the four arithmetic operations for intervals of real numbers and prove the existence of fixed points without using Brouwer's fixed point theorem. The proof is simple. However, the assumptions are stronger than necessary for the application of Brouwer's theorem.

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1 Introduction

The following result is well known.

Theorem 1.1 (Fixed point Theorem of Brouwer) *Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be continuous, $D \neq \emptyset$, convex and compact, and let the range $W(f;D) = \{f(x) | x \in D\}$ be contained in D, $W(f;D) \subseteq D$. Then f has a fixed point in D.*

Therefore, if for a given interval vector $[x] \subseteq D$ the interval arithmetic evaluation $f([x])$ of f over $[x]$ is contained in $[x]$, then $W(f;[x]) \subseteq f([x]) \subseteq [x]$ and by Theorem 1.1, f has a fixed point in $[x]$.

The preceding application of Theorem 1.1 can be found repeatedly in algorithms which are based on interval arithmetic tools. See [4], for example.

Unfortunately, the proof of Theorem 1.1 is far from trivial. See e.g. [4], where one can find several different proofs. In this paper we present a possibility of avoiding

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Brouwer's fixed point theorem. The price we have to pay is that we have to assume a condition stronger than $f([x]) \subseteq [x]$.

The paper is organized as follows: In Section 2 we repeat the notation usually used in interval arithmetic, repeat the basic arithmetic operations and recall some basic properties. After that we introduce four new operations $\bar{*} \in \{\bar{+}, \bar{-}, \bar{\times}, \bar{/}\}$, which especially have the property that the midpoint of the result is obtained by performing the operations for the midpoints of the operands. We show that the new operations are inclusion monotone. In Section 3 we prove a theorem which guarantees the existence of a fixed point. In Section 4 we consider the case that the components of f contain also nonrational terms. The final Section 5 contains a numerical example.

2 Notation

Let $\mathbb{R}$ denote the set of real numbers, $* \in \{+, -, \times, /\}$ one of the four binary operations of the real numbers and denote by $I\mathbb{R}$ the set of real closed and bounded intervals. The elements of $I\mathbb{R}$ are denoted by $[a], [b], \ldots$. Therefore,

$$[a] = \{a \in \mathbb{R}, a_1 \leq a \leq a_2, a_1, a_2 \in \mathbb{R}\},$$

for example. We also write $[a] = [a_1, a_2]$. The four arithmetic operations in $I\mathbb{R}$ are defined by the corresponding set operations

$$[a] * [b] := \{a * b \mid a \in [a], b \in [b]\} \tag{1}$$

where $* \in \{+, -, \times, /\}$. For the division, we assume $0 \notin [b]$. For (1), we obtain the following results expressed in terms of the bounds of $[a]$ and $[b]$. Let $[a] = [a_1, a_2]$, $[b] = [b_1, b_2]$. Then

$$\left.\begin{array}{l}
[a] + [b] = [a_1 + b_1, a_2 + b_2]\,, \\
[a] - [b] = [a_1 - b_2, a_2 - b_1]\,, \\
[a] \times [b] = [\min\{a_1b_1, a_2b_2, a_1b_2, a_2b_1\}, \max\{a_1b_1, a_2b_2, a_1b_2, a_2b_1\}]\,. \\
\text{Using} \\
\quad 1 \,/\, [b] = [\frac{1}{b_2}, \frac{1}{b_1}] \quad \text{for } 0 \notin [b], \\
\text{we obtain} \\
\quad [a] \,/\, [b] = [a] \times (1 \,/\, [b]) \quad \text{for } 0 \notin [b].
\end{array}\right\} \tag{2}$$

Because of definition (1), the so-called *inclusion monotonicity* holds:
If

$$[a]^1 \subseteq [a]^2 \quad \text{and} \quad [b]^1 \subseteq [b]^2 \tag{3}$$

then

$$[a]^1 * [b]^1 \subseteq [a]^2 * [b]^2 \quad \text{for } * \in \{+, -, \times, /\}, \tag{4}$$

where we assume $0 \notin [b]^2$ for division. Using this property repeatedly, it follows that

$$f(x) \in f([a]), \quad x \in [a], \tag{5}$$

if f is a rational function and $f([a])$ denotes the so-called interval arithmetic evaluation of f over $[a]$, which is obtained by replacing x by $[a]$ and the operations by the corresponding interval operations (provided this leads to a well-defined result $f([a])$). (5) is called *inclusion property*.

If f is a rational function of several variables, analogous properties hold.

If $[x] = [x_1, x_2] \in I\mathbb{R}$ then

$$m = \frac{x_1 + x_2}{2}, \tag{6}$$

and

$$r = \frac{x_2 - x_1}{2} \tag{7}$$

are called *center* (or *midpoint*) and *radius* of $[x]$, respectively. Hence, $[x]$ can also be written as

$$[x] = [m - r, m + r] .$$

With the notation

$$\langle m, r \rangle := [m - r, m + r] \quad \text{for } m \in \mathbb{R},\ r \geq 0,$$

we have

$$[x] = \langle m, r \rangle .$$

We now introduce a *midpoint-radius interval arithmetic* as follows:

1. If $*$ is one of the binary interval operations in $\{+, -, \times, /\}$ defined in (2) and

   $$\langle m, r \rangle * \langle n, s \rangle = [c_1, c_2]$$

   then

   $$[c_1, c_2] = [m * n - \alpha, m * n + \beta]$$

   where

   $$\alpha = m * n - c_1 \geq 0, \quad \beta = c_2 - m * n \geq 0.$$

   The *tightest inclusion* of $[c_1, c_2]$ by an interval with midpoint $m * n$ is then

   $$\langle m, r \rangle \,\bar{*}\, \langle n, s \rangle := \left\langle m * n, r' \right\rangle \text{ with } r' = \max\{\alpha, \beta\}. \tag{8}$$

2. If f is a real valued continous function defined on $\langle m, r \rangle$ with range $W(f; \langle m, r \rangle) = [c_1, c_2]$ then

   $$[c_1, c_2] = [f(m) - \alpha, f(m) + \beta]$$

   where

   $$\alpha = f(m) - c_1 \geq 0, \quad \beta = c_2 - f(m) \geq 0.$$

   The *tightest inclusion* of $[c_1, c_2]$ by an interval with midpoint $f(m)$ is then

   $$\bar{f}\, \langle m, r \rangle := \left\langle f(m), r' \right\rangle \text{ with } r' = \max\{\alpha, \beta\}. \tag{9}$$

Remark 1 *Obviously*

$$\langle m,r\rangle * \langle n,s\rangle \subseteq \langle m,r\rangle \,\bar{*}\, \langle n,s\rangle \;\; \textit{for} * \in \{+,-,\times,/\}$$

and

$$f(\langle m,r\rangle) \subseteq \bar{f}\,\langle m,r\rangle$$

with always one identical bound of the compared intervals.

If $* = \pm$, it is easily seen, that

$$\langle m,r\rangle \,\bar{\pm}\, \langle n,s\rangle = \langle m \pm n, r+s\rangle \;\; (= \langle m,r\rangle \pm \langle n,s\rangle)\,. \tag{10}$$

If $* = \times$ we have

$$\langle m,r\rangle \times \langle n,s\rangle = [m-r, m+r] \times [n-s, n+s] = [c_1, c_2]$$

where

$$\begin{aligned} c_1 &= \min\Big\{(m-r)(n-s), (m+r)(n+s), (m-r)(n+s), (m+r)(n-s)\Big\} \\ &= mn + \min\Big\{-ms-rn+rs, ms+rn+rs, ms-rn-rs, -ms+rn-rs\Big\} \\ &= mn - \max\Big\{ms+rn-rs, -ms-rn-rs, -ms+rn+rs, ms-rn+rs\Big\} \end{aligned}$$

and

$$c_2 = mn + \max\Big\{-ms-rn+rs, ms+rn+rs, ms-rn-rs, -ms+rn-rs\Big\}.$$

Therefore

$$\begin{aligned} r' = \max\Big\{ &\max\Big\{ms+rn-rs, -ms-rn-rs, -ms+rn+rs, ms-rn+rs\Big\}, \\ &\max\Big\{-ms-rn+rs, ms+rn+rs, ms-rn-rs, -ms+rn-rs\Big\}\Big\} \\ = |m|s &+ |n|r + rs. \end{aligned}$$

Thus,

$$\langle m,r\rangle \,\bar{\times}\, \langle n,s\rangle = \langle mn, |m|s+|n|r+rs\rangle\,. \tag{11}$$

If $* = /$ and $0 \notin \langle n,s\rangle$, we have by (2)

$$\begin{aligned} &\langle m,r\rangle \,/\, \langle n,s\rangle = \langle m,r\rangle \times (1 \,/\, \langle n,s\rangle) = [m-r, m+r] \times [\frac{1}{n+s}, \frac{1}{n-s}] \\ &= \left[\min\left\{\frac{m-r}{n+s}, \frac{m+r}{n+s}, \frac{m-r}{n-s}, \frac{m+r}{n-s}\right\}, \max\left\{\frac{m-r}{n+s}, \frac{m+r}{n+s}, \frac{m-r}{n-s}, \frac{m+r}{n-s}\right\}\right]. \end{aligned}$$

Hence, by (8),

$$\langle m,r\rangle \,\bar{/}\, \langle n,s\rangle = \left\langle \frac{m}{n}, r'\right\rangle, \tag{12}$$

where $r' = \max\{\alpha, \beta\}$ with

$$\alpha = \frac{m}{n} - \min\left\{\frac{m-r}{n+s}, \frac{m+r}{n+s}, \frac{m-r}{n-s}, \frac{m+r}{n-s}\right\}$$
$$\beta = \max\left\{\frac{m-r}{n+s}, \frac{m+r}{n+s}, \frac{m-r}{n-s}, \frac{m+r}{n-s}\right\} - \frac{m}{n}.$$

α can also be written as

$$\begin{aligned}\alpha &= \max\left\{\frac{m}{n} - \frac{m-r}{n+s}, \frac{m}{n} - \frac{m+r}{n+s}, \frac{m}{n} - \frac{m-r}{n-s}, \frac{m}{n} - \frac{m+r}{n-s}\right\}\\ &= \max\left\{\frac{nr}{n(n+s)} + m\frac{s}{n(n+s)}, \frac{-nr}{n(n+s)} + m\frac{s}{n(n+s)},\right.\\ &\qquad\left.\frac{nr}{n(n-s)} - m\frac{s}{n(n-s)}, \frac{-nr}{n(n-s)} - m\frac{s}{n(n-s)}\right\}\\ &= \max\left\{\frac{|n|r}{n(n+s)} + m\frac{s}{n(n+s)}, \frac{|n|r}{n(n-s)} - m\frac{s}{n(n-s)}\right\}.\end{aligned}$$

Analogously we get

$$\beta = \max\left\{\frac{|n|r}{n(n+s)} - m\frac{s}{n(n+s)}, \frac{|n|r}{n(n-s)} + m\frac{s}{n(n-s)}\right\}.$$

Hence

$$\begin{aligned}r' &= \max\left\{\frac{|n|r}{n(n+s)} + |m|\frac{s}{n(n+s)}, \frac{|n|r}{n(n-s)} + |m|\frac{s}{n(n-s)}\right\}\\ &= (|n|r + |m|s)\max\left\{\frac{1}{n(n+s)}, \frac{1}{n(n-s)}\right\}.\end{aligned} \tag{13}$$

If $f(x) = 1\,/\,x,\ x \neq 0$, then for $\langle m, r\rangle = \langle 1, 0\rangle$ and $0 \notin \langle n, s\rangle$ we get

$$\overline{f}\,\langle n, s\rangle = \left\langle 1\,/\,n, s'\right\rangle \quad \text{with} \quad s' = s\max\left\{\frac{1}{n(n+s)}, \frac{1}{n(n-s)}\right\}. \tag{14}$$

Remark 2 *The operations $\overline{+}, \overline{-}$ and $\overline{\times}$ appear also in the midpoint-radius interval arithmetic which was studied in [1] even in the complex case. However,*

$$1\,/\,\langle n, s\rangle = \langle\frac{n}{n^2 - s^2}, \frac{s}{n^2 - s^2}\rangle$$

was used for inversion instead of $\overline{f}\,\langle n, s\rangle$ given just before the remark in (14)*.*

If f is constant on $\langle m, r\rangle$, $f(x) = c, x \in \langle m, r\rangle$, then obviously $\overline{f}\,\langle m, r\rangle = \langle c, 0\rangle = [c, c]$.

If $f(x) = x, x \in \langle m, r\rangle$, then $\overline{f}\,\langle m, r\rangle = \langle m, r\rangle$.

The inclusion property of the introduced operations follows easily from Remark 1. For the proof of the announced fixed point theorem it is crucial, however, that also the *inclusion monotonicity* holds, i.e.

$$\langle m_1, r_1\rangle \subseteq \langle m_2, r_2\rangle \quad \text{and} \quad \langle n_1, s_1\rangle \subseteq \langle n_2, s_2\rangle$$

imply

$$\langle m_1, r_1 \rangle \,\bar{*}\, \langle n_1, s_1 \rangle \subseteq \langle m_2, r_2 \rangle \,\bar{*}\, \langle n_2, s_2 \rangle \,.$$

For $* = \overline{\pm}$ inclusion monotonicity holds since it holds for $* = \pm$. This is obviously true also for $f(x) = c$, $x \in \langle m, r \rangle$, or $f(x) = x$, $x \in \langle m, r \rangle$.

Inclusion monotonicity for $* = \overline{\times}$ was shown in [1], even in the complex case.

In passing we note that the operations $\bar{*} \in \{\overline{+}, \overline{-}, \overline{\times}\}$ have already been mentioned in [2].

Let us now consider $\bar{f}$ for $f(x) = 1 \,/\, x$, $x \neq 0$.

We show first, that

$$\langle m, r \rangle \subseteq \langle n, s \rangle$$

implies

$$\bar{f}\,\langle m, r \rangle \subseteq \bar{f}\,\langle n, s \rangle \;\text{ for } f(x) = 1 \,/\, x, \quad 0 \notin \langle n, s \rangle \,.$$

Let $[a] = \langle m, r \rangle \subseteq [b] = \langle n, s \rangle$ be given with $0 \notin [b]$, i.e. $b_1 = n - s > 0$ or $b_2 = n + s < 0$.

If $n - s > 0$ then

$$\bar{f}\,\langle n, s \rangle = \langle \frac{1}{n}, \frac{s}{m(n-s)} \rangle = \left[\frac{3b_1 - b_2}{(b_1 + b_2)b_1}, \frac{1}{b_1} \right]$$

and, since also $m - r > 0$,

$$\bar{f}\,\langle m, r \rangle = \left[\frac{3a_1 - a_2}{(a_1 + a_2)a_1}, \frac{1}{a_1} \right].$$

Consequently $\bar{f}\,\langle m, r \rangle \subseteq \bar{f}\,\langle n, s \rangle$ iff

$$\frac{3b_1 - b_2}{(b_1 + b_2)b_1} \leq \frac{3a_1 - a_2}{(a_1 + a_2)a_1} \quad \text{and} \quad \frac{1}{a_1} \leq \frac{1}{b_1}.$$

The second inequality follows from $0 < b_1 \leq a_1$. In order to verify the first one, we study the function

$$h(x, y) = \frac{3x - y}{(x + y)x}, \quad 0 < x < y.$$

Since the partial derivatives of h satisfy

$$\begin{aligned} h_x(x, y) &= \frac{-3x^2 + 2xy + y^2}{((x + y)x)^2} > \frac{-3x^2 + 2x^2 + x^2}{((x + y)x)^2} = 0, \\ h_y(x, y) &= -\frac{4x^2}{((x + y)x)^2} < 0, \end{aligned}$$

we can conclude

$$h(b_1, b_2) \leq h(a_1, b_2) \leq h(a_1, a_2),$$

which shows the desired result $h(b_1, b_2) \leq h(a_1, a_2)$ in case $a_1 < a_2$. If $a_1 = a_2 (= m)$ then obviously $\bar{f}\,\langle m, r \rangle = \langle f(m), 0 \rangle \subseteq \bar{f}\,\langle n, s \rangle$.

The case $n + s < 0$ can be treated by symmetry.

Remark 3 *If* $0 \notin \langle n, s\rangle$*, then, similarly to the relation* $[a] \,/\, [b] = [a] * (1 \,/\, [b])$ *for* $0 \notin [b]$ *from* (2)*, we have*

$$\langle m, r\rangle \,\bar{/}\, \langle n, s\rangle = \langle m, r\rangle \,\bar{\times} (1 \bar{/}\, \langle n, s\rangle). \tag{15}$$

This can be seen as follows:

$$\langle m, r\rangle \,\bar{\times} (1 \bar{/}\, \langle n, s\rangle) = \langle m, r\rangle \,\bar{\times} \langle \frac{1}{n}, s'\rangle,$$

where, by (14)*,*

$$s' = s \max\left\{\frac{1}{n(n+s)}, \frac{1}{n(n-s)}\right\}.$$

Hence

$$\langle m, r\rangle \,\bar{\times} (1 \bar{/}\, \langle n, s\rangle) = \langle \frac{m}{n}, |m|s' + \frac{1}{|n|}r + rs'\rangle.$$

Therefore, by (13)*, it remains to show that*

$$|m|s' + \frac{1}{|n|}r + rs' = (|n|r + |m|s) \max\left\{\frac{1}{n(n+s)}, \frac{1}{n(n-s)}\right\},$$

i.e.

$$\frac{1}{|n|}r + rs' = |n|r \max\left\{\frac{1}{n(n+s)}, \frac{1}{n(n-s)}\right\},$$

i.e.

$$\frac{1}{|n|} + s' = |n| \max\left\{\frac{1}{n(n+s)}, \frac{1}{n(n-s)}\right\}.$$

The last equation is in the case $n - s > 0$ *identical with the identity*

$$\frac{1}{n} + \frac{s}{n(n-s)} = n\frac{1}{n(n-s)},$$

in the case $n + s < 0$ *identical with the identity*

$$-\frac{1}{n} + \frac{s}{n(n+s)} = -n\frac{1}{n(n+s)},$$

which completes the proof of the remark.

Since we have shown above that $\bar{f}\,\langle n, s\rangle$ is inclusion monotone for $f(x) = 1 \,/\, x$, $x \neq 0$ and since the multiplication "$\bar{\times}$" is inclusion monotone it follows by (15) that the division "$\bar{/}$" is inclusion monotone.

3 Fixed Points

The following result holds.

Theorem 3.1 *Let*

$$f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$$

be a rational function. Assume that the interval vector $[x] = ([x]_i)$ *is contained in* D*. Assume that the generalized interval arithmetic evaluation* $\overline{f}([x])$ *exists and*

$$\overline{f}([x]) \subseteq [x]. \tag{16}$$

Then f *has a fixed point* $x^* \in [x]$*.*

Remark 4 *The generalized interval arithmetic evaluation* $\overline{f}\,([x])$ *of* f *is defined componentwise by replacing in all components of* f *the variables* $x_1, \ldots, x_n$ *by the intervals* $[x]_1, \ldots, [x]_n$ *and evaluating the resulting expression by using the operations "*$\overline{*}$*". The result is an interval vector.*

Note the different meaning of the bar "$\bar{\ }$" in (9) and (16), respectively. Especially, in the case $n = 1$ we have for a given $[x] = \langle m, r\rangle$ the relation $\overline{f}\langle m, r\rangle \subseteq \overline{f}([x])$, where the equality sign only holds in special cases.

Remark 5 *We can even show, that* $x^* \in \overline{f}\,([x])$*.*

Proof of Theorem 3.1: Let $[x]^0 := [x]$ and consider the iteration method

$$[x]^{k+1} = \overline{f}([x]^k), \quad k = 0, 1, 2, \ldots \ . \tag{17}$$

Because of (17) and (16) we have

$$[x]^1 = \overline{f}([x]^0) \subseteq [x]^0.$$

Since the operations "$\overline{*}$" are inclusion monotone it follows by mathematical induction that

$$[x]^{k+1} = \overline{f}([x]^k) \subseteq [x]^k, \quad k = 0, 1, 2, \ldots \ .$$

Therefore, one obtains a nested sequence $\left\{[x]^k\right\}_{k=0}^{\infty}$ which is convergent to some interval vector $[x]^* = \left[(x^1)^*, (x^2)^*\right]$.

We consider the sequence $\left\{m^k\right\}_{k=0}^{\infty}$ with $m^k = m([x]^k)$ (where the components of $m([x]^k)$ are defined componentwise). Since for the four operations "$\overline{*}$" it holds $m([a]\overline{*}[b]) = m[a] * m[b]$ we obtain from the iteration (17)

$$m([x]^{k+1}) = f\left(m[x]^k\right), \quad k = 0, 1, 2, \ldots \ . \tag{18}$$

Since the components of $m([x]^k)$ are continuous functions of the interval bounds $(x^1)^k$ and $(x^2)^k$ of $[x]^k$, they are also convergent. Denote the limit by m^*. Then it holds

$$\begin{aligned} m^* = \lim_{k\to\infty} m([x]^k) &= \lim_{k\to\infty} \frac{1}{2}\left((x^1)^k + (x^2)^k\right) \\ &= \frac{1}{2}\left((x^1)^* + (x^2)^*\right). \end{aligned}$$

Since f is continuous we obtain from (18)

$$m^* = f(m^*),$$

which means that m^* is a fixed point of f in $[x]^0$. $\square$

Remark 6 *The preceding proof of the existence of a fixed point m^* can also be used to approximate m^* under the assumption $\overline{f}([x]^0) \subseteq [x]^0$: Perform the iteration method (17) and compute the sequence of midpoints by forming in each step the center $m([x]^k)$ of $[x]^k$. This sequence $\left\{m([x]^k)\right\}_{k=0}^{\infty}$ is converging to m^*.*

Remark 7 *If $\overline{f}\left([x]^0\right) \subseteq [x]^0$, then it follows that all fixed points of f in $[x]^0$ are even contained in $[x]^* \subseteq [x]^0$ (and not only in $\overline{f}([x]^0)$ as stated in Remark 5). This can be seen as follows: Assume m^* is a fixed point of f in $[x]^0$. Then by the inclusion property*

$$m^* = f(m^*) \in \overline{f}\left([x]^0\right) = [x]^1 \subseteq [x]^0.$$

By mathematical induction it follows

$$m^* = f(m^*) \in \overline{f}\left([x]^k\right) = [x]^{k+1},$$

that is m^* is contained in all iterates and therefore also in the limit $[x]^*$. Since m^* was an arbitrary fixed point the remark is proved.

In concluding we remark that checking (16) and performing (17) on a computer, rounding errors have to be taken into account in order to guarantee the validity of Theorem 3.1. For details see, e.g., [3].

4 Generalizations

Since $\overline{f}\,\langle m, r\rangle$ is defined for any real valued continous function f on $\langle m, r\rangle$, it is possible to define $\overline{f}\,\langle m, r\rangle$ especially for the standard functions $\mathrm{sqrt}(\cdot), \sin(\cdot), \exp(\cdot), \ln(\cdot), \ldots,$ provided the interval $\langle m, r\rangle$ is contained in the respective domain. In order to extend the proof of Theorem 3.1 for mappings using also such functions, it must be guaranteed, however, that inclusion monotonicity holds for the corresponding enclosure $\overline{f}\,\langle m, r\rangle$.

Analyzing the proof of inclusion monotonicity of $\overline{f}\,\langle n, s\rangle$ in the case $f(x) = 1\,/\,x$, $x > 0$, one observes that it is based on the fact, that f is differentiable, decreasing and convex on $\langle n, s\rangle$. This observation is a guide for proving that inclusion monotonicity also holds for $\overline{f}\,\langle n, s\rangle$, if f is differentiable on $\langle n, s\rangle$, decreasing and concave, or increasing and convex, or increasing and concave, respectively. Therefore, inclusion monotonicity holds e.g. for

$$\begin{aligned}
&\overline{\mathrm{sqrt}}\,\langle n, s\rangle\,, \quad n - s > 0;\\
&\overline{\exp}\,\langle n, s\rangle\,;\\
&\overline{\ln}\,\langle n, s\rangle\,, \quad n - s > 0;\\
&\overline{\cos}\,\langle n, s\rangle\,, \quad \langle n, s\rangle \subseteq \langle \frac{\pi}{4}, \frac{\pi}{4}\rangle = [0, \frac{\pi}{2}].
\end{aligned}$$

Let us now demonstrate the proof for a differentiable function f on $\langle n, s\rangle$ which is increasing and concave (like $\mathrm{sqrt}(\cdot)$ on $\langle n, s\rangle$, $n - s > 0$).

Given $[a] = \langle m, r\rangle \subseteq [b] = \langle n, s\rangle$. Then $f([b]) = [f(b_1), f(b_2)]$ and

$$\overline{f}\,\langle n, s\rangle = \left\langle f(n), r'\right\rangle \quad \text{with} \quad r' = \max\{f(n) - f(b_1), f(b_2) - f(n)\}.$$

Since f is concave,

$$f(n) \geq \frac{f(b_1) + f(b_2)}{2},$$

from which it follows that

$$f(n) - f(b_1) \geq f(b_2) - f(n)$$

and therefore

$$\bar{f}\left\langle n, s \right\rangle = \left[f(b_1), 2f\left(\frac{b_1 + b_2}{2}\right) - f(b_1) \right].$$

Analogously,

$$\bar{f}\left\langle m, r \right\rangle = \left[f(a_1), 2f\left(\frac{a_1 + a_2}{2}\right) - f(a_1) \right].$$

Hence

$$\bar{f}\left\langle m, r \right\rangle \subseteq \bar{f}\left\langle n, s \right\rangle$$

holds iff

$$f(b_1) \leq f(a_1) \quad \text{and} \quad 2f\left(\frac{a_1 + a_2}{2}\right) - f(a_1) \leq 2f\left(\frac{b_1 + b_2}{2}\right) - f(b_1).$$

The first inequality holds since f is increasing. In order to verify the second one, we study the function

$$h(x, y) = 2f\left(\frac{x + y}{2}\right) - f(x), \quad b_1 \leq x < y \leq b_2.$$

For the partial derivatives we obtain

$$h_x(x, y) = f'\left(\frac{x + y}{2}\right) - f'(x), \quad h_y(x, y) = f'\left(\frac{x + y}{2}\right).$$

Since f is concave, f' is decreasing. Hence $h_x(x, y) \leq 0$ for $x < y$. Since f is increasing, $h_y(x, y) \geq 0$. Therefore,

$$h(a_1, a_2) \leq h(b_1, a_2) \leq h(b_1, b_2)$$

shows the desired inequality $h(a_1, a_2) \leq h(b_1, b_2)$ in the case $a_1 < a_2$. If $a_1 = a_2 (= m)$ then obviously

$$\bar{f}\left\langle m, r \right\rangle = \left\langle f(m), 0 \right\rangle \subseteq \bar{f}\left\langle n, s \right\rangle.$$

We point out, that inclusion monotonicity cannot be concluded from monotonicity of f on $\langle n, s \rangle$ alone, as the example

$$[a] = [0, \frac{\pi}{2}] \subseteq [b] = [0, \pi],$$
$$\overline{\cos}[a] = [0, \sqrt{2}] \not\subseteq \overline{\cos}[b] = [-1, 1]$$

shows.

For the square function $\operatorname{sqr}(x) = x^2, x \in \mathbb{R}$, we have

$$\overline{\operatorname{sqr}}\left\langle m, r \right\rangle = \left\langle m^2, r'' \right\rangle, \quad \text{where} \quad r'' = \max\{\alpha, \beta\},$$
$$\alpha = \max\{x^2 | x \in \langle m, r \rangle\} - m^2, \quad \beta = m^2 - \min\{x^2 | x \in \langle m, r \rangle\}.$$

Case 1: $m - r \geq 0$. Then

$$\alpha = (m+r)^2 - m^2 = 2mr + r^2$$
$$\beta = m^2 - (m-r)^2 = 2mr - r^2.$$

Hence $r'' = \alpha = 2mr + r^2$ and $\overline{\text{sqr}}\,\langle m, r\rangle = \left\langle m^2, 2mr + r^2\right\rangle$.

Case 2: $m + r \leq 0$. Then

$$\alpha = (m-r)^2 - m^2 = -2mr + r^2,$$
$$\beta = m^2 - (m+r)^2 = -2mr - r^2.$$

Hence $r'' = \alpha = -2mr + r^2$ and $\overline{\text{sqr}}\,\langle m, r\rangle = \left\langle m^2, -2mr + r^2\right\rangle$.

Case 3: $m - r < 0 < m + r (\Leftrightarrow |m| < r)$. Then

$$\max\{x^2 | x \in \langle m, r\rangle\} = \max\{(m-r)^2, (m+r)^2\}$$
$$= \max\{m^2 - 2mr + r^2, m^2 + 2mr + r^2\} = m^2 + 2|m|r + r^2.$$
$$\min\{x^2 | x \in \langle m, r\rangle\} = 0.$$

Hence $\alpha = 2|m|r + r^2$, $\beta = m^2 < r^2$, $r'' = \alpha = 2|m|r + r^2$.

But this shows that we have

$$\overline{\text{sqr}}\,\langle m, r\rangle = \left\langle m^2, 2|m|r + r^2\right\rangle \tag{19}$$

in all three cases.

Since we have also

$$\langle m, r\rangle \mathbin{\overline{\times}} \langle m, r\rangle = \langle mm, |m|r + |m|r + rr\rangle = \left\langle m^2, 2|m|r + r^2\right\rangle,$$

the relation

$$\overline{\text{sqr}}\,\langle m, r\rangle = \langle m, r\rangle \mathbin{\overline{\times}} \langle m, r\rangle \tag{20}$$

holds, showing especially inclusion monotonicity of $\overline{\text{sqr}}$. This is somewhat surprising, because sqr is of course convex, but not monotone.

In fact, convexity alone generally is not sufficient to guarantee inclusion monotonicity as the following example shows:

For the convex function f defined by $f(x) = 1\,/\,x + x\,/\,16, x > 0$, and the intervals $[a] = [2, 6] \subseteq [b] = [2, 7]$, we get

$$\bar{f}[a] = [0.375, 0.625] \not\subseteq \bar{f}[b] = [0.38194..., 0.625].$$

In passing we note that (19) and (20) can be generalized for $f(x) = x^k$, $k = 1, 2, \ldots$ as follows:

$$\bar{f}\langle m, r\rangle = \langle m^k, (|m| + r)^k - |m|^k\rangle, \tag{19a}$$

$$\langle m, r\rangle \overline{\times} \cdots \overline{\times} \langle m, r\rangle = \bar{f}\langle m, r\rangle. \tag{20a}$$

5 Numerical Example

In order to demonstrate the applied arithmetic and to show that (16) is a fulfillable condition, let us consider a numerical example (from [5], page 85).

In this example $f : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$f(x) = \begin{pmatrix} 1 + 0.05(1 - x_1)x_2 - 0.025x_2^2 \\ -0.5 + 0.025(1 - x_1)^2 + 0.02x_2^2 \end{pmatrix}, x = (x_1, x_2)^T \in \mathbb{R}^2.$$

$x^0 = (1, -0.5)^T$ is an approximate fixed point of f.

We ask now, whether there is an $\varepsilon > 0$, such that (16) holds for $[x] = (\langle 1, \varepsilon\rangle, \langle -0.5, \varepsilon\rangle)^T$, i.e. such that

$$\langle 1, 0\rangle \overline{+} \langle 0.05, 0\rangle \overline{\times}(\langle 1, 0\rangle \overline{-} \langle 1, \varepsilon\rangle)\overline{\times} \langle -0.5, \varepsilon\rangle \overline{-} \langle 0.025, 0\rangle \overline{\times}\overline{\mathrm{sqr}} \langle -0.5, \varepsilon\rangle \subseteq \langle 1, \varepsilon\rangle,$$

and

$$\langle -0.5, 0\rangle \overline{+} \langle 0.025, 0\rangle \overline{\times}\overline{\mathrm{sqr}}(\langle 1, 0\rangle \overline{-} \langle 1, \varepsilon\rangle)\overline{+} \langle 0.02, 0\rangle \overline{\times}\overline{\mathrm{sqr}} \langle -0.5, \varepsilon\rangle \subseteq \langle -0.5, \varepsilon\rangle.$$

After carrying out the overlined operations, we get the conditions

$$\langle 0.99375, (0.05 + 0.075\varepsilon)\varepsilon\rangle \subseteq \langle 1, \varepsilon\rangle$$

and

$$\langle -0.495, (0.02 + 0.045\varepsilon)\varepsilon\rangle \subseteq \langle -0.5, \varepsilon\rangle.$$

Now, as one can easily check, these conditions are satisfied, i.e. (16) holds, e.g. for each ε between 0.01 and 0.03. For $\varepsilon = 0.01$, the conditions read

$$\langle 0.99375, 0.0005075\rangle = [0.9932425, 0.9942575] \subseteq [0.99, 1.01] = \langle 1, 0.01\rangle,$$

and

$$\langle -0.495, 0.0002045\rangle = [-0.4952045, -0.4947955] \subseteq [-0.51, -0.49] = \langle -0.5, 0.01\rangle.$$

Note, that $(0.99375, -0.495)^T = f((1, -0.5)^T)$, since the applied arithmetic preserves midpoints.

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